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ON SEMIGROUP VALUED ADDITIVE EXHAUSTIVE SET FUNCTIONS

Endre Pap
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

1. INTRODUCTION

Using the functional from |4| defined on a semigroup we shall introduce the notions of the variation and of the semivariation for semigroup valued additive set functions. We shall prove that these notions have the usual properties.

By the Diagonal Theorem from |4| we shall prove, in quite an elementary way, two theorems on the uniform boundedness of a family of regular additive exhaustive set functions on a Borel algebra (i.e. on a σ-algebra generated by the open subsets of a compact Hausdorff topological space). Namely, if the domain is not a σ-algebra, the Nikodym boundedness theorem, in general, is not true (see |2|). But there are known uniform boundedness theorems in which the initial boundedness conditions are given on some subfamilies which are not σ-algebras (for example in |5|). In the stated theorems this subfamily is the family of open sets.

2. THE VARIATION AND SEMIVARIATION

Let X be a commutative semigroup with the neutral element 0 endowed with a triangle functional f such that

$$f(x+y) \leq f(x) + f(y),$$

$$f(x+y) \geq f(x) - f(y) \quad \text{and}$$

$$f(0) = 0 \qquad (\text{see } |4|).$$

REMARK 1.

The topology of a uniform semigroup is generated by a family of special pseudometrics - H.Weber |7|. If follows from this construction, that there exists a family of triangle functions on every uniform semigroup - E.Pap |6|.

Let Σ be an algebra of subsets of the set S. A set function μ defined on the algebra Σ with the values in X is additive if whenever E_1 and E_2 are disjoint members of Σ then

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

We suppose that there exists $B \in \Sigma$ such that $\mu(B) = 0$. It follows from additivity, by the preceding assumption, that $\mu(\emptyset) = 0$.

A set function $\mu : \Sigma \to X$ is said to be exhaustive (f-exhaustive) if $\lim_{n\to\infty} f(\mu(E_n)) = 0$ for every sequence (E_n) of disjoint members of Σ.

The variation of an additive set function $\mu : \Sigma \to X$ is the nonnegative set function $|\mu|$ defined for $E \in \Sigma$ by

$$|\mu|(E) = \sup_{\pi} \sum_{A\in\pi} f(\mu(A))$$

where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of Σ.

Obviously $f(\mu(E)) \leq |\mu|(E)$ for each $E \in \Sigma$ and $|\mu|(\emptyset) = 0$.

PROPOSITION 1. *If $\mu : \Sigma \to X$ is an additive set function on an algebra Σ, then $|\mu|$ is also additive on Σ.*

P r o o f. Let E_1 and E_2 be two disjoint sets from Σ. If $|\mu|(E_1 \cup E_2) < \infty$, then for $\varepsilon > 0$ we can choose finite systems $\{A_i^1\}$ and $\{A_i^2\}$ of pairwise disjoint sets from Σ such that $A_i^1 \subset E_1$ and $A_i^2 \subset E_2$ and

$$|\mu|(E_i) \leq \sum_j f(\mu(A_j^i)) + \varepsilon \quad \text{for} \quad i = 1,2.$$

Then we have

$$|\mu|(E_1) + |\mu|(E_2) \leq \sum_j f(\mu(A_j^1)) + \sum_j f(\mu(A_j^2)) + 2\varepsilon \leq$$

$$\leq |\mu|(E_1 \cup E_2) + 2\varepsilon .$$

Since $\varepsilon > 0$ was arbitrary we obtain the finite superadditivity of $|\mu|$, i.e.

$$(1) \qquad |\mu|(E_1) + |\mu|(E_2) \leq |\mu|(E_1 \cup E_2) .$$

On the other hand, we take partition $\{A_j\}$ of $E_1 \cup E_2$ into a finite number of pairwise disjoint members of Σ. We take $A_j^1 = E_1 \cap A_j$ and $A_j^2 = E_2 \cap A_j$. Then we have

$$\sum_j f(\mu(A_j)) \leq \sum_j f(\mu(A_j^1)) + \sum_j f(\mu(A_j^2)) \leq |\mu|(E_1) + |\mu|(E_2) .$$

Hence

$$|\mu|(E_1 \cup E_2) \leq |\mu|(E_1) + |\mu|(E_2) .$$

By the preceding inequality and (1) we obtain the additivity of $|\mu|$. The last inequality implies also the additivity in the case $|\mu|(E_1 \cup E_2) = \infty$.

REMARK 2.

$|\mu|$ is superadditive in general, i.e.

$$|\mu|\left(\bigcup_{i \in I} E_i\right) \geq \sum_{i \in I} |\mu|(E_i) .$$

This follows in the same way as in |3| , p.35.

An additive set function $\mu : \Sigma \to X$ is said to be of bounded variation if $|\mu|(S) < \infty$.

If μ is of bounded variation then it is also bounded (f-bounded), but in general the converse is not true (for example $X = L^{\infty}[0,1]$, Σ is a σ-algebra of Lebesgue measurable subsets of $[0,1]$ and $\mu(E) = \chi_E$).

The semivariation of an additive set function $\mu : \Sigma \to X$ is the nonnegative set function $\|\mu\|$ defined for $E \in \Sigma$ by

$$\|\mu\|(E) = \sup\{f(\mu(H)) | H \in \Sigma,\ H \subset E\}.$$

Obviously:

(i) $\|\mu\|(\emptyset) = 0$ (ii) $f(\mu(E)) \leq \|\mu\|(E)$ for each $E \in \Sigma$ (iii) $\|\mu\|(E) \leq |\mu|(B)$ for each $E \in \Sigma$, (iv) $\|\mu\|(F) \leq \|\mu\|(E)$ for $F \subset E$ and $F, E \in \Sigma$.

PROPOSITION 2. *If* $\mu : \Sigma \to X$ *is an additive set function on an algebra* Σ, *then* $\|\mu\|$ *is subadditive on* Σ, *i.e.*

$$\|\mu\|(E_1 \cup E_2) \leq \|\mu\|(E_1) + \|\mu\|(E_2)$$

for every $E_1, E_2 \in \Sigma$ *such that* $E_1 \cap E_2 = \emptyset$.

P r o o f. Let E_1 and E_2 be two disjoint sets from Σ. Let H be an arbitrary subset of $E_1 \cup E_2$. Then we have

$$f(\mu(H)) = f(\mu(H \setminus E_1 \cup H \setminus E_2)) \leq f(\mu(H \setminus E_1)) + $$
$$+ f(\mu(H \setminus E_2)) \leq \|\mu\|(E_1) + \|\mu\|(E_2).$$

Hence the desired inequality follows.

An additive set function $\mu : \Sigma \to X$ is said to be of bounded semivariation if $\|\mu\|(S) < \infty$. By the definition we obtain directly that the semivariation of the additive set function $\mu : \Sigma \to X$ is bounded iff μ is bounded.

REMARK 3.

The semivariation is usually defined for a Banach space valued additive set function as

$$\sup\{|x^*\mu|(E)\,|\,x^* \in X^*,\ \|x^*\| \leq 1\}$$

where $|x^*\mu|$ is the variation of $x^*\mu$ (see |2|), or, equivalently as

$$\sup\{\|\sum_{A_m \in \pi} \varepsilon_n \mu(A_n)\|\}$$

where the supremum is taken over all partitions π of E into finitely many disjoint members of Σ and over all finite collections $\{\varepsilon_n\}$ satisfyng $|\varepsilon_n| \leq 1$ (|3|, p.51 or Proposition 11.a. from |2|, p.4. Our definition of the semivariation, in the case of Banach space valued additive set functions, is equivalent, in the sense of the norm, to the usual one (Proposition 11.b. p.4. from |2|).

PROPOSITION 3. *An additive set function* $\mu : \Sigma \to X$ *is exhaustive iff* $\lim_{n\to\infty} \|\mu\|(E_n) = 0$ *for every sequence* (E_n) *of disjoint members of* Σ.

P r o o f. Let μ be an exhaustive additive set function. Suppose that $\|\mu\|$ does not satisfy the proposition. Then there exist $\delta > 0$, a sequence (E_n) of disjoint members of Σ for which

$$\|\mu\|(E_n) \geq \delta$$

holds. For each n and ε such that $\delta > \varepsilon > 0$ there is $H_n \in \Sigma$ such that $H_n \subset E_n$ and

$$\|\mu\|(E_n) \leq f(\mu(H_n)) + \varepsilon.$$

Then the sequence (H_n) consists of disjoint members of Σ such that

$$f(\mu(H_n)) \geq \delta - \varepsilon > 0$$

holds for each n, which contradict the exhaustivity of μ. The converse implication follows from the inequality

$$f(\mu(E)) \leq \|\mu\|(E) \quad \text{for each } E \in \Sigma.$$

THEOREM 1. *An exhaustive additive set function* $\mu:\Sigma \to X$ *on an algebra* Σ *of subsets of* S *is of bounded semivariation.*

P r o o f. Let us suppose that the theorem is not true. Then there exists $H_1 \in \Sigma$ such that

$$(2) \qquad f(\mu(H_1)) \geq 1 + 2f(\mu(S))$$

Since

$$f(\mu(H_1)) - f(\mu(S \setminus H_1)) \leq f(\mu(H_1) + \mu(S \setminus H_1)) = f(\mu(S)) \ ,$$

we obtain

$$f(\mu(S \setminus H_1)) \geq 1 \ .$$

Since

$$\|\mu\|(S) \leq \|\mu\|(H_1) + \|\mu\|(S \setminus H_1) \ ,$$

either $\|\mu\|(H_1)$ or $\|\mu\|(S \setminus H_1)$ is infinite. If H_1 is with unbounded semivariation we put $E_1 = H_1$, otherwise we take $E_1 = S \setminus H_1$ we have that E_1 has the unbounded semivariation and $f(\mu(E_1)) \geq 1$. Now we take E_1 instead of S and repeat the preceding procedure.

We obtain $E_2 \in \Sigma$ and $E_2 \subset E_1$ such that the semivariation of E_2 is unbounded and $f(\mu(E_2)) \geq 2$. By iterating we construct a nonincreasing sequence (E_n) from Σ such that the semivariation of E_n is unbounded for each n and

$$(3) \qquad f(\mu(E_n)) \geq n \ .$$

Let $F_n = E_n \setminus E_{n+1}$. Since $\mu(E_n \setminus E_{n+1}) + \mu(E_{n+1}) = \mu(E_n)$ we obtain

$$f(\mu(F_n)) \geq f(\mu(E_{n+1})) - f(\mu(E_n)).$$

Then, by (3), it follows that $f(\mu(F_n)) \geq 1$ for some disjoint subsequence of (F_n) from Σ, which contradicts the exhaustive of μ.

3. UNIFORM BOUNDEDNESS ON BOREL SETS

In this section Σ is a σ-algebra of subsets of a set S. An additive set function $\mu : \mathcal{B} \to X$ defined on the collection $\mathcal{B}$ of Borel sets of a compact Hausdorff topological space T is regular on a Borel set B if for every $\varepsilon > 0$ there exists a compact set $K \subset B$ and an open set $O \supset B$ such that

$$\|\mu\|(O \setminus K) < \varepsilon .$$

A set function $\mu : \Sigma \to X$ is <u>f-superadditive</u> if for every sequence (E_n) of disjoint members of a σ-algebra Σ

$$\lim_{n\to\infty} f(\mu(\bigcup_{i=1}^{n} E_i)) \leq f(\mu(\bigcup_{i=1}^{\infty} E_i))$$

holds.We shall use the following.

DIAGONAL THEOREM. *(|4|). If* $x_{ij} \in X$ $(i,j \in N)$ *and*

$$\lim_{j\to\infty} f(x_{ij}) = 0 \quad \textit{for} \quad i=1,2,\ldots ,$$

then there exist an infinite set I *of positive integers and a subset* J *(finite or infinite) of* I *such that, for all* $i \in I$, *we have*

$$\sum_{j\in J} f(x_{ij}) < \infty$$

$$f(\sum_{j\in J} x_{ij}) \geq \frac{1}{2} f(x_{ii}) .$$

(where J *is infinite)*

$$f(\sum_{j\in J} x_{ij}) \stackrel{def}{=} \lim_{n\to\infty} f(\sum_{s=1}^{n} x_{ij_s}),$$

(j_s) is the increasing sequence of all elements of J.

THEOREM 2. *Let $(\mu_\alpha)_{\alpha\in A}$ be a family of* X*-valued regular additive and f-superadditive set functions defined on the Borel subsets of a compact Hausdorff topological space* T. *If*

$$\{\mu_\alpha(O) \mid \alpha \in A\}$$

is bounded on every open set O*, then the family* $(\mu_\alpha)_{\alpha\in A}$ *is uniformly bounded, i.e.*

$$\sup\{f(\mu_\alpha(B)) \mid \alpha \in A,\ B \in \mathcal{B}\} < \infty \quad .$$

P r o o f. Since the general case can be reduced to the proof for the sequence, we take a sequence (μ_n) instead of the family.

To prove the uniform boundedeness of (μ_n), it suffices to prove that every point in T belongs to an open set O on which

$$\sup_{n,A} \{f(\mu_n(A)),\quad n \in N,\ A \subset O\} < \infty$$

Suppose that this is not true. Then there exists $x \in T$ such that $\sup_{n\in N} \|\mu_n\|(O)$ is not bounded for every open set O such that $x \in O$.

Let O be an open set which contains x. Then there exists a Borel set $B \subset O$ and $n_1 \in N$ such that

$$(4) \qquad f(\mu_{n_1}(B)) > 4 + 2 \sup_n f(\mu_n(\{x\})) \quad .$$

Since μ_{n_1} is regular, there exists a compact set $K_o \subset B$ such that

$$(5) \qquad \|\mu_{n_1}\|(O \setminus K_o) < 1.$$

Then, by the inequalities

$$f(\mu_{n_1}(K_o)) + f(\mu_{n_1}(B \quad K_o)) \geq f(\mu_{n_1}(B)),$$

$$f(\mu_{n_1}(B \setminus K_o)) \leq \|\mu_{n_1}\|(B \setminus K_o) \leq \|\mu_{n_1}\|(O \setminus K_o),$$

(4) and (5) we obtain

$$f(\mu_{n_1}(K_o)) > 3 + 2 \sup_n f(\mu_n(\{x\})) \quad .$$

Let $K_1 = K_o \cup \{x\}$. Then we have

$$f(\mu_{n_1}(K_1)) > 3 + \sup_n f(\mu_n(\{x\})) \quad .$$

By the regularity of μ_{n_1} there exists an open set U such that $O \supset U \supset K_1$ and

$$\|\mu_{n_1}\| (U \setminus K_1) < 1 .$$

Hence, by the inequality

$$f(\mu_{n_1}(U)) \geq f(\mu_{n_1}(K_1)) \dot{-} f(\mu_{n_1}(U \setminus K_1)),$$

we obtain

(6) $$f(\mu_{n_1}(U)) > 2 + \sup_n f(\mu_n(\{x\})).$$

We find, again by regularity, an open set W such that $\{x\} \subset W \subset U$ and $\|\mu_{n_1}\| (W \setminus \{x\}) < 1$. Let H be an open set such that $x \in H \subset \bar{H} \subset W$. Then we have

(7) $$f(\mu_{n_1}(\bar{H})) \leq \|\mu_{n_1}\| (\bar{H} \setminus \{x\}) + f(\mu_{n_1}(\{x\})) \leq$$

$$\leq \|\mu_{n_1}\| (W \setminus \{x\}) + f(\mu_{n_1}(\{x\}) < 1 + \sup_n f(\mu_n(\{x\})).$$ If we

take that $E_1 = U \setminus \bar{H}$, then we have $E_1 \subset 0$, $E_1 \cap H = \emptyset$. By the inequality

$$f(\mu_{n_1}(E_1)) + f(\mu_{n_1}(\bar{H})) \geq f(\mu_{n_1}(U)) ,$$

(6) and (7), we obtain

$$f(\mu_{n_1}(E_1)) > 1.$$

Using the same procedure, the fact that $x \in H$ and that the sequence (μ_n) is unbounded on H, and taking in the inequality (4) $5+2\sup_n f(\mu_n(\{x\}))$ instead of $4+2 \sup_n f(\mu_n(\{x\}))$, we obtain an integer $n_2 > n_1$, open sets $E_2, H_1 \subset H$ such that $E_1 \cap E_2 = \emptyset$, $E_2 \cap H_1 = \emptyset$, $x \in H_1$ and $f(\mu_{n_2}(E_2)) > 2$. Continuing, we obtain a sequence of positive integers (n_k) and a sequence of pairwise disjoint open sets (E_k) such that

(8) $$f(\mu_{n_k}(E_k)) > k \quad \text{for every } k \in N.$$

Now we take $x_{ij} = \mu_{n_i}(E_j)$ for $i,j \in N$. Then we have for each $i \in N$

$$\lim_{j\to\infty} f(x_{ij}) = 0.$$

Then there exist , by the Diagonal Theorem, an infinite set $I\subset N$ and its subset J such that

$$f(\sum_{j\in J} x_{ij}) \geq \frac{1}{2} f(x_{ii}) \tag{9}$$

for each $i \in I$. Since μ_{n_i} are additive and f-superadditive we have

$$\lim_{n\to\infty} f(\sum_{s=1}^{n} \mu_{n_i}(E_{j_s})) = \lim_{n\to\infty} f(\mu_{n_i}(\bigcup_{s=1}^{n} E_{j_s}))\leq f(\mu_{n_i}(\bigcup_{s=1}^{\infty} E_{j_s}))$$

(if J is finite we need only the first equality). Then by (9) and (8) we obtain

$$f(\mu_{n_i}(\bigcup_{j\in J} E_j)) > \frac{i}{2}$$

for each $i \in I$, which is a contradiction with the boundedness of (μ_n) on every open set.

Using the main part of the proof of Theorem 2 we obtain a uniform boundedness - type theorem connected with the variation.

THEOREM 3. *Let $(\mu_\alpha)_{\alpha\in A}$ be a family of X-valued regular additive set functions defined on the Borel subsets of a compact Hausdorff topological space T. If μ_α is of bounded variation on every open set, for each $\alpha \in A$, then the family is uniformly bounded.*

P r o o f. The proof is verbatum the same as up to (9). If J is infinite, we have

$$\lim_{n\to\infty} f(\sum_{s=1}^{n} \mu_{n_i}(E_{j_s}) \leq \lim_{n\to\infty} \sum_{s=1}^{n} f(\mu_{n_i}(E_{j_s}))\leq$$

$$\leq \lim_{n\to\infty} \sum_{s=1}^{n} |\mu_{n_i}|(E_{j_s}) \leq |\mu_{n_i}|(\bigcup_{j\in J} E_j)$$

where (j_s) is the increasing sequence of all elements of J. By (9) and (8) we have

$$|\mu_{n_i}|(\bigcup_{j\in J} E_j) > \frac{i}{2}$$

for every $i \in I$ which is a contradiction.

REFERENCES

|1| J.K.Brooks, *On a Theorem of Dieudonne, Advances in Math. 36, 165-168, (1980).*

|2| J.Diestel, J.J.Uhl, *Vector Measures, AMS, Math. surveys 15, 1977.*

|3| N.Dinculeanu, *Vector Measures, Pergamon Press, 1967.*

|4| E.Pap, *A generalization of the diagonal theorem on a block-matrix,Mat. vesnik 11 (26), 1974, 66-71.*

|5| E.Pap, *Uniform boundendness of family of triangle semigroup valued set functions, Zbornik radova PMF u Novom Sadu, 10, 1980.*

|6| E.Pap, *Funkcionalna analiza (English summary), Institute of Mathematics, Novi Sad, 1982.*

|7| H.Weber, *Fortsetzung von Massen mit Werten in uniformen Helbgruppen, Arch. der Math. 27, (1976), 412-423.*

REZIME

O ADITIVNOJ EKSHAUSTIVNOJ FUNKCIJI SKUPA SA VREDNOSTIMA U POLUGRUPI

U radu se ispituje funkcija skupa μ sa vrednostima u komutativnoj polugrupi X sa neutralnim elementom 0 i trougaonom funkcionelom f, za koju važi: $f(x+y) \leq f(x) + f(y)$, $f(x+y) \geq f(x)-f(y)$ i $f(0) = 0$. Izdvajaju se aditivne i ekshaustivne funkcije skupa na algebri Σ skupova. Uvode se ne-negativne funkcije skupa, pridružene funkciji skupa sa vrednostima u polugrupi μ, varijacija $|\mu|$ i poluvarijacija $\|\mu\|$.

Pomoću Dijagonalne teoreme iz |4|, dokazuju se dve teoteme o uniformnoj ograničenosti:

Teorema 2. Neka je $(\mu_\alpha)_{\alpha\in A}$ familija X vrednosnih regularnih aditivnih i f-superaditivnih funkcija skupa definisanih na Borelovim podskupovima kompaktnog Hausdorfovog topološkog prostora T. Ako je $\{\mu_\alpha(O) | \alpha \in A\}$ ograničeno na svakom otvorenom skupu O,tada je familija $(\mu_\alpha)_{\alpha\in A}$ i uniformno ograničena na svim Borelovim skupovima.

Teorema 3. Neka je $(\mu_\alpha)_{\alpha \in A}$ familija X vrednosnih regularnih aditivnih funkcija skupa definisanih na Borelovim podskupovima kompaktnog Hausdorfovog topološkog prostora T. Ako je μ_α ograničene varijacije na svakom otvorenom skupu, za svako $\alpha \in A$, tada je familija uniformno ograničena.