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UNBOUNDED SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION

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Consider an initial value problem

(1) $$y'' = f(x)y^{\lambda} \quad ; \quad y(a) = b_o \ , \quad y(a) = b_1$$

where $\lambda > 1$, $a > 0$, and $f(x)$ is continuous and positive for $x > a$.

By an unbounded solution of (1) is meant a solution possessing the property

$$y(x) \to \infty \quad \text{for} \quad x \to \omega - 0$$

where ω is some positive real number.

Such solutions were first considered by R.H.Fowler in 1931, |1| , for $f(x) = x^r$, r real.

J.Karamata and V.Marić |2| proved the existence of unbounded solutions of (1) for any $b_o > 0$, $b_1 > 0$ provided that

(2) $$f(x) \geq m > 0 \ , \quad x > 0 \ .$$

If (2) holds for $a \leq x \leq b$ only, then the above result is still valid but only for some appropriately chosen values of b_1.

The interest for unbounded solutions has been renewed by E.Hille (cf. |3|, |4|, |5|) who proved the existence of these for the Thomas-Fermi ($f(x) = x^{-1/2}$; $\lambda = 1{,}5$) and Emden ($f(x) = x^{1-m}$, $\lambda = m$, $m > 1$) equations occurring in various applications, with $b_o \geq 0$, $b_1 \geq 0$ and $b_o + b_1 > 0$.

Hille's result is generalized by L.E.Bobisud |6| for the equation

(3) $$y'' = f(x)g(y)$$

where f,g are continuous and positive for $x>0$, $y>0$; if

(4) $$\int_a^\infty \left(\int_a^x g(u)du \right)^{-1/2} dx < \infty$$

and

$$f'(x)f^{-3/2}(x) \to 0 \quad \text{for} \quad x \to \infty .$$

A special case $b_1 = 0$ ($b_o > 0$) is studied by S.B.Eliason |7| .

V.Marić and M.Skendžić |8| proved an existence theorem for unbounded solutions of the equation (3) for $b_o > 0$, $b_1 \geq 0$ and under less restrictive conditions on $f(x)$:

$$f(x) \geq h(x) \quad , \quad \int^\infty \sqrt{h(x)}dx = \infty$$

where $h(x)$ is a positive, continuous, decreasing function which tends to zero for $x \to \infty$; $g(u)$ satisfies (4).

In the present we shall prove the existence of unbounded solutions for $y'(a) = b_1 < 0$. To do this we shall first prove two lemmas.

LEMMA 1. *Let* $y(x)$, $Y(x)$ *be non-negative solutions of the initial value problems*

$$y'' = yf(x,y) \quad , \quad y(a) = Y(a)$$

$$Y'' = YF(x,Y) \quad , \quad y'(a) < Y'(a)$$

where f,F *are continuous functions such that* $0 < f(u,v) < F(u,v)$ *for* $a \leq u \leq b$ *and* $v > 0$, *and let* $F(u,v)$ *be a strictly increasing function of* v *for each* u.

Then $y(x) < Y(x)$ *and* $y'(x) < Y'(x)$ *for* $a < x \leq b$.

P r o o f. For $a \leq x \leq b$ we have

$$y(x) = y(a) + (x-a)y'(a) + \int_a^x (x-s)y(s)f(s,y(s))ds$$

$$y(x) = y(a) + (x-a)y'(a) + \int_a^x (x-s)y(s)f(s,y\ (s))ds$$

$$Y(x) = Y(a) + (x-a)Y'(a) + \int_a^x (x-s)Y(s)F(s,Y(s))ds$$

So that

$$(5)\quad Y(x)-y(x)=(x-a)(Y'(a)-y'(a)) + \int_a^x (x-s)(Y(s)F(s,Y(s)) - \\ - y(s)F(s,y(s)))ds\ .$$

Since $y'(a) < Y'(a)$ there exists $\varepsilon > 0$ such that $Y(x)-y(x) > 0$ for $a < x < a+\varepsilon$. Suppose $Y(x) \leq y(x)$ for some $x \in (a,b]$. Then there exists $c \in (a,b]$ such that $Y(x)-y(x) > 0$ for $a < x < c$ and $Y(c)=y(c)$. Since $y(x)$ and $Y(x)$ are non-negative, $Y(x)F(x,Y(x))-y(x)f(x,y(x)) > 0$ for $a < x < c$. Putting $x=c$, in (5) the left hand side is zero and the right hand side is positive, which is a contradiction. Thus $y(x) < Y(x)$ for $a < x \leq b$. Moreover

$$y'(x) = y'\ (a) + \int_a^x y(s)f(s,y(s))ds$$

$$Y'(x) = Y'(a) + \int_a^x Y(s)F(s,Y(s))ds.$$

Hence $y'(x) < Y'(x)$ for $a \leq x \leq b$.

LEMMA 2. *Suppose* $y_1(x)$ *and* $y_2(x)$ *are two positive solutions of*

$$y'' = yF(x,y)\ ;\quad y_1(a) = y_2(a);\quad y_1'(a) < y_2'(a)$$

where F *satisfies the same conditions as in Lemma 1. Then* $y_2(x) - y_1(x) \geq (x-a)(y_2'(a)-y_1'(a))$ *for* $x \geq a$.

P r o o f. According to Lemma 1. $y_2(x) = y_1(x)$. Therefore

$$y_2''(x)-y_1''(x) = y_2(x)F(x,y_2(x)) - y_1(x)F(x,y_1(x)) \geq 0.$$

So $y_2'(x)-y_1'(x) = y_2'(a)-y_1'(a)$ and hence, after integration over $[a,x]$, the proof is finished.

The consequence of this Lemma is that the positive, decreasing solution defined for all $x > a$ of the initial value problem

$$y'' = yF(x,y)\ ;\quad y(a) = b_o\ ;\quad y'(a) = b_1 < 0$$

(provided that it exists) is unique and all other are not bounded (including both i.e. such that $y(x) \to \infty$, $x \to \omega - 0$, or $y(x) \to \infty$, $x \to \infty$).

The existence of such solutions is proved by P.K.Wong |9| as follows: there exists a positive decreasing solution $y(x)$ which tends, for $x \to \infty$, to a positive constant iff there is a $\beta > 0$ such that $\int^{\infty} xF(x,\beta)dx$ converges. The divergence of that integral for all $\beta > 0$ is, on the other hand, a neccessary and sufficient condition for the existence of a solution tending to zero for $x \to \infty$.

When $y'(a) > b_1$, corresponding solutions, if they are defined for all $x \geq a$, are not slower than a linear function. By some restriction on F, existence of solutions such that $y \sim kx$, $x \to \infty$ was proved by P.Waltman |10|.

Denote by $y(x)$ the positive solutions of the initial value problems

$$(6) \qquad y'' = yF(x,y) \quad ; \quad y(a) = b_0 \quad ; \quad y'(a) = b_1$$

and let b_∞ stand for the initial slope of the unique positive decreasing bounded solution of (6) (defined for all $x \geq a$). Then we prove the following.

THEOREM *If the initial value problem*

$$Y'' = YF(x,Y) \quad ; \quad Y(x_0) = b_0 \quad ; \quad Y'(x_0) = 0$$

has unbounded solutions for each x_0, *then any positive solution* $y(x)$ *of (6), satisfying* $y'(a) > b_\infty$, *is unbounded too.*

P r o o f. First take $b_\infty < b_1 < 0$, then according to Lemma 2 for the solution $y(x)$ there exists a point $x_0 > a$ such that $y(x_0) = y(a) = b_0$. Since the solution is convex, $y'(x_0) > Y'(x_0) = 0$ and then, by Lemma 1, is unbounded.

If $b_1 \geq 0$, one may take $x_0 = a$ and repeat the argument.

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REZIME

NEOGRANIČENA REŠENJA NELINEARNE DIFERENCIJALNE JEDNAČINE

U ovom radu je pokazano kada početni problem

$$y'' = yF(x,y)\ ,\ y(a) = b_0,\quad y'(a) = b_1$$

ima neograničeno rešenje za $b_1 < 0$.