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A COMMON FIXED POINT THEOREM OF A FAMILY OF MAPPINGS IN PROBABILISTIC LOCALLY CONVEX SPACES

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In this paper a theorem which gives the necessary and sufficient condition for the existence of a unique fixed point for the mappings S, T and A (defined in Theorem 2) is proved.

First we shall give some definitions and notations which we shall use later.

DEFINITION 1 |1| . *Let* X *be a linear space over the real or complex field* K *and for every* i *in the index set* I *there is a function* $F^i : X \to \Delta^+$, *where* Δ^+ *is the family of distribution functions* F *such that* $F(0) = 0$. *We shall denote* $F^i(p)$ *by* F^i_p $(i \in I,\ p \in X)$.

$(X, \{F^i\}_{i\in I}, t)$ *is called a locally convex probabilistic space (LCP-space) if and only if for each* $i \in I$ *the following conditions are satisfied:*

1. $F^i_o = H$, *where* $H(\varepsilon) = \begin{cases} 0, & \varepsilon \le 0 \\ 1, & \varepsilon > 0 \end{cases}$,
2. $F^i_{\lambda p}(\varepsilon) = F^i_p\left(\frac{\varepsilon}{|\lambda|}\right)$ *for every* $\lambda \in K,\ p \in X,\ \varepsilon > 0, (\lambda \neq 0)$
3. $F^i_{p+q}(\varepsilon_1+\varepsilon_2) \ge t(F^i_p(\varepsilon_1),\ F^i_q(\varepsilon_2))$ *for every* $p, q \in X$,

$\varepsilon_1 > 0,\ \varepsilon_2 > 0$, *where the mapping* $t : [0,1] \times [0,1] \to [0,1]$ *is a* T-*norm* |2| .

The (ε,λ)- topology is introduced by the following definition.

DEFINITION 2 |1|. *A net* $\{p_d\}_{d\in D}$ *converges to* $\mathcal{O}$ *if and only if for each* $i \in I$, *every* $\varepsilon > 0$ *and every* $\lambda \in (0,1)$ *there exists* $d_o \in D$ *such that*

$$F^i_{p_d}(\varepsilon) \geq 1-\lambda \quad \text{if} \quad d \geq d_o .$$

In what follows we suppose that

$$F^i_p = H \quad \text{for each } i \in I \text{ if and only if } \quad p = \mathcal{O}.$$

In a similar way one can introduce the notion of a Cauchy sequence and the notion of completeness.

Some fixed point theorems in probabilistic spaces are proved in |1|, |2|.

THEOREM 1 |3|. *Let* S *and* T *be continuous mappings of a complete normed space* $(X, \|\cdot\|)$ *into istelf. Then* S *and* T *have a common fixed point in* X *if and only if there exists a continuous mapping* A *of* X *into* $SX \cap TX$ *which commutes with* S *and* T *and satisfies the inequality*

$$\|Ax-Ay\| \leq \alpha \|Sx-Ty\|$$

for every $x,y \in X$, *where* $0 < \alpha < 1$. S ,T *and* A *then have a unique common fixed point.*

THEOREM 2. $(X,\{F^i\}_{i\in I},t)$ *be a sequentially complete probabilitic locally convex space with continuous* T*-norm* t *and* S *and* T *be continuous mappings of* X *into* X.

The mappings S *and* T *have a unique common fixed point in* X *if and only if there exists a continuous mapping* $A: X \to SX \cap TX$ *which commutes with* S *and* T *so that* AX *is a probabilistic bounded subset of* X *and satisfies the following conditions:*

1. For every $i \in I$ *there exists* $q(i) > 0$ *and* $f(i) \in I$ *such that*

$$F^{i}_{Ax-Ay}(\varepsilon) \geq F^{f(i)}_{Tx-Sy}\left(\frac{\varepsilon}{q(i)}\right)$$

for every $\varepsilon > 0$ *and every* $x,y \in X$.

2. For every $i \in I$ *there exist numbers* $n_i \in N$ *and* $Q(i) \in (0,1)$ *such that*

$$q(f^{n}(i)) \leq Q(i) < 1$$

for every $n > n_i$.

3. For every $i \in I$ *there exists* $g(i) \in I$ *such that*

$$F^{f^{n}(i)}_{x}(\varepsilon) \geq F^{g(i)}_{x}(\varepsilon)$$

for every $\varepsilon > 0$*, every* $x \in X$ *and every* $n \in N$.

Then there exists one and only one element $x^{*} \in X$ *such that* Ax^{*} *is the unique common fixed point for the mappings* A, S *and* T .

P r o o f. First, let us prove the necessity of the conditions 1, 2 and 3. Let $z \in X$ be such an element that

$$z = Az = Sz = Tz \ .$$

The mapping A is defined by $Ax = z$ for all $x \in X$.

The mapping A commutes with S and T since

$$A(Sx) = Ay = z \ , \qquad S(Ax) = Sz = z$$

and

$$A(Tx) = Av = z, \qquad T(Ax) = Tz = z.$$

Condition 1 will be satisfied because

$$F^{i}_{Ax-Ay}(\varepsilon) = F^{i}_{z-z}(\varepsilon) = 1 \geq F^{i}_{Tx-Sy}\left(\frac{\varepsilon}{q(i)}\right)$$

for all $x,y \in X$, i.e. $f(i) = i$ for all $i \in I$ and $q(i)$ is any element from $(0,1)$.

Since $f(i) = i$, contition 3 is also satisfied, i.e.

$$F^{f^{n}(i)}_{x}(\varepsilon) = F^{i}_{x}(\varepsilon)$$

for all $x \in X$ and all $\varepsilon > 0$. We have that $g(i) = i$, $i \in I$.

Now, we shall prove that conditions 1,2 and 3 are sufficient. We form a sequence $\{x_n\}_{n \in N \cup \{0\}}$ such that

$$Ax_{2n-2} = Sx_{2n-1}, \qquad Ax_{2n-1} = Tx_{2n}$$

for all $n \in N$. Such a sequence always exists because $AX \subset TX \cap SX$.

First, we prove that the sequence $\{Ax_n\}_{n \in N \cup \{0\}}$ is a Cauchy sequence. It is easy to see that

$$F^{i}_{Ax_{2n}-Ax_{2n-1}}(\varepsilon) \geq F^{f(i)}_{Tx_{2n}-Sx_{2n-1}}\left(\frac{\varepsilon}{q(i)}\right) =$$

$$= F^{f(i)}_{Ax_{2n-1}-Ax_{2n-2}}\left(\frac{\varepsilon}{q(i)}\right) \geq F^{f^2(i)}_{Tx_{2n-2}-Sx_{2n-1}}\left(\frac{\varepsilon}{q(i)q(f(i))}\right) \geq$$

$$\geq F^{f^2(i)}_{Ax_{2n-2}-Ax_{2n-3}}\left(\frac{\varepsilon}{q(i)q(f(i))}\right) \geq \cdots \geq$$

$$\geq F^{f^{2n-1}(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon}{\prod_{s=o}^{2n-2} q(f^s(i))}\right)$$

for all $\varepsilon > 0$, $i \in I$, $n \in N$. Also we have that

$$F^{i}_{Ax_{2n+1}-Ax_{2n}}(\varepsilon) = F^{i}_{Ax_{2n}-Ax_{2n+1}}(\varepsilon) \geq \cdots \geq$$

$$\geq F^{f^{2n}(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon}{\prod_{s=o}^{2n-1} q(f^s(i))}\right)$$

for all $\varepsilon > 0$, $i \in I$, $n \in N$. So for every $n \in N$ we have

$$F^{i}_{Ax_n-Ax_{n-1}}(\varepsilon) \geq F^{f^{n-1}(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon}{\prod_{s=o}^{n-2} q(f^s(i))}\right)$$

for all $\varepsilon > 0$, $i \in I$.

We see that for all $m > k$

$$F^{i}_{Ax_{2k}-Ax_{2m+1}}(\varepsilon) \geq F^{f(i)}_{Sx_{2m+1}-Tx_{2k}}\left(\frac{\varepsilon}{q(i)}\right) =$$

$$= F^{f(i)}_{Ax_{2m}-Ax_{2k-1}}\left(\frac{\varepsilon}{q(i)}\right) \geq \dots \geq F^{f^{2k}(i)}_{Ax_{o}-Ax_{2m+1-2k}}\left(\frac{\varepsilon}{\prod_{s=o}^{2k-1} q(f^{s}(i))}\right).$$

If $2k > 2m+1$ then we have

$$F^{i}_{Ax_{2k}-Ax_{2m+1}}(\varepsilon) \geq F^{f(i)}_{Ax_{2k-1}-Ax_{2m}}\left(\frac{\varepsilon}{q(i)}\right) \geq \dots \geq$$

$$\geq F^{f^{2m+1}(i)}_{Ax_{2k-2m-1}-Ax_{o}}\left(\frac{\varepsilon}{\prod_{s=o}^{2m} q(f^{s}(i))}\right)$$

for every $\varepsilon > 0$.

From the last two inequalities we can prove easily that for $n = 2k$, $p = 2m+1$ or for $n = 2k+1$, $p = 2m+1$ the following is satisfied

$$F^{i}_{Ax_{n+p}-Ax}(\varepsilon) \geq F^{f^{n}(i)}_{Ax_{o}-Ax_{p}}\left(\frac{\varepsilon}{\prod_{s=o}^{n-1} q(f^{s}(i))}\right).$$

When $n = 2k$, $p = 2m$ or $n = 2k+1$, $p = 2m$ we have

$$F^{i}_{Ax_{n+p}-Ax_{n}}(\varepsilon) = F^{i}_{Ax_{n+p}-Ax_{n+1}+Ax_{n+1}-Ax_{n}}\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) \geq$$

$$\geq t\left(F^{i}_{Ax_{n+p}-Ax_{n+1}}\left(\frac{\varepsilon}{2}\right), F^{i}_{Ax_{n+1}-Ax_{n}}\left(\frac{\varepsilon}{2}\right)\right) \geq \dots \geq$$

$$\geq t\left(F^{f^{n+1}(i)}_{Ax_{p-1}-Ax_{o}}\left(\frac{\varepsilon}{2\prod_{s=o}^{n} q(f^{s}(i))}\right), F^{f^{n}(i)}_{Ax_{1}-Ax_{o}}\left(\frac{\varepsilon}{2\prod_{s=o}^{n-1} q(f^{s}(i))}\right)\right).$$

Now we shall prove that for every $\varepsilon > 0$, $\lambda \in (0,1)$ and $i \in I$ there exists $N(i,\varepsilon,\lambda)$ such that

$$F^i_{Ax_{n+p}-Ax_n}(\varepsilon) > 1-\lambda$$

for every $n \geq N(i,\varepsilon,\lambda)$ and $p \in N$.

The set AX is probabilistic bounded which means that

$$\sup_{\varepsilon} D^i_{AX}(\varepsilon) = 1$$

for all $i \in I$ and so for each $\lambda \in (0,1)$, there exists $\varepsilon_i(\lambda) > 0$ such that

$$D^i_{AX}(\varepsilon_i(\lambda)) > 1-\lambda.$$

Since

$$D^i_{AX}(\varepsilon_i) = \sup_{\delta<\varepsilon_i} \inf_{u,v\in AX} F^i_{u-v}(\delta)$$

it follows that:

$$\sup_{\delta<\varepsilon_i} \inf_{u,v\in AX} F^i_{u-v}(\delta) > 1-\lambda,$$

and we have that

$$\inf_{u,v\in AX} F^i_{u-v}(\varepsilon_i(\lambda)) \geq \inf_{u,v\in AX} F^i_{u-v}(\delta)$$

which implies

$$\inf_{u,v\in AX} F^i_{u-v}(\varepsilon_i(\lambda)) \geq \sup_{\delta<\varepsilon_i} \inf_{u,v\in AX} F^i_{u-v}(\delta) > 1-\lambda .$$

Let for every $i \in I$: $\varepsilon(i) = \dfrac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))}$, for every $\varepsilon > 0$.

Let $n > n_i$. Then we have

$$\frac{\varepsilon}{\prod_{s=0}^{n-1} q(f^s(i))} = \frac{\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))q(f^{n_i+1}(i))\dots q(f^{n-1}(i))} \geq$$

$$\geq \frac{\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i)Q^{n-n_i-1}(i)} = \frac{\varepsilon(i)}{Q^n(i)} .$$

The mapping t is continuous and since $t(1,1)=1$, for $\lambda \in (0,1)$ there exists $r \in (0,1)$ so that for all $x \geq r$ and all $y \geq r$ we have $t(x,y) > 1-\lambda$. Let $\varepsilon_i(r)$ be such that

$$\inf_{u,v \in AX} F^{g(i)}_{u,v}(\varepsilon_i(r)) > r \; .$$

Suppose that $r > 1-\lambda$. Further, let $n(i,r) \in N$ be such that

$$\frac{\varepsilon(i)}{2Q^n(i)} \geq \varepsilon_i(r) \; ,$$

for every $n \geq n(i,r)$.

If $n = 2k$, $p = 2m+1$ or $n = 2k+1$, $p = 2m+1$ and $n \geq n(i,r) + n_i$ it follows that:

$$F^i_{Ax_{n+p}-Ax_n}(\varepsilon) \geq F^{f^n(i)}_{Ax_o-Ax_p}\left(\frac{\varepsilon(i)}{Q^n(i)}\right)$$

where $\varepsilon(i) = \dfrac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=o}^{n_i} q(f^s(i))}$.

Furthermore

$$F^{f^n(i)}_{Ax_o-Ax_p}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \geq F^{f^n(i)}_{Ax_o-Ax_p}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right) \geq F^{g(i)}_{Ax_o-Ax_p}(\varepsilon_i(r)) > r > 1-\lambda .$$

If $n = 2k$, $p = 2m$ or $n = 2k+1$, $p = 2m$ and $n > n(i,r)+n_i$ then

$$F^i_{Ax_{n+p}-Ax_n}(\varepsilon) \geq t\left(F^{g(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right), F^{g(i)}_{Ax_{p-1}-Ax_o}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right)\right) \geq$$

$$\geq t(r,r) > 1-\lambda \; .$$

So we have proved that the sequence $\{Ax_n\}_{n \in N}$ is a Cauchy sequence and since X is sequentially complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} Ax_n = x^* \; .$$

According to the construction of the sequence $\{x_n\}_{n\in N}$ we have

$$\lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n} = x^* .$$

Since

$$\lim_{n\to\infty} Ax_n = x^*$$

we have the following implications

$$\lim_{n\to\infty} Ax_{2n} = x^* \Rightarrow T(\lim_{n\to\infty} Ax_{2n}) = Tx^* \Rightarrow \lim_{n\to\infty} TAx_{2n} = Tx^* \Rightarrow$$

$$\Rightarrow A(\lim_{n\to\infty} Tx_{2n}) = Tx^* \Rightarrow A(\lim_{n\to\infty} Tx_{2n-1}) = Tx^* \Rightarrow Ax^* = Tx^*.$$

Also

$$\lim_{n\to\infty} Ax_{2n-1} = x^* \Rightarrow S(\lim_{n\to\infty} Ax_{2n-1}) = Sx^* \Rightarrow \lim_{n\to\infty} SAx_{2n-1} =$$

$$= Sx^* \Rightarrow A(\lim_{n\to\infty} Sx_{2n-1}) = Sx^* \Rightarrow A(\lim_{n\to\infty} Sx_{2n-2}) =$$

$$= Sx^* \Rightarrow Ax^* = Sx^* ,$$

and it follows that

$$Ax^* = Sx^* = Tx^* .$$

Now we shall show that Ax^* is a fixed point of the mapping A.

$$F^{i}_{Ax^*-A^2x^*}(\varepsilon) \geq F^{f(i)}_{Sx^*-TAx^*}\left(\frac{\varepsilon}{q(i)}\right) = F^{f(i)}_{Sx^*-ATx^*}\left(\frac{\varepsilon}{q(i)}\right) =$$

$$= F^{f(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon}{q(i)}\right) \geq \dots \geq F^{f^n(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon}{\prod_{s=0}^{n-1} q(f^s(i))}\right) \geq$$

$$\geq F^{f^n(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \geq F^{g(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon(i)}{Q^n(i)}\right)$$

where $\varepsilon(i) = \dfrac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))}$, $(n > n_i)$.

We have that

$$F^{g(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon}{Q^n(i)}\right) \to 1 \text{ , for } n \to \infty$$

and so

$$F^{i}_{Ax^*-A^2x^*}(\varepsilon) = 1$$

for all $\varepsilon > 0$ and all $i \in I$. This implies that

$$Ax^* = A^2x^* = A(Ax^*) \text{ ,}$$

i.e. Ax^* is a fixed point of the mapping A.

It is easy to see that Ax^* is a fixed point of the mappings S and T

$$S(Ax^*) = A(Sx^*) = A(Ax^*) = Ax^*$$

and

$$T(Ax^*) = A(Tx^*) = A(Ax^*) = Ax^* \text{ .}$$

Now we shall show that Ax^* is a unique fixed point of the mappings A, S and T. If we suppose that x_1 is another common fixed point of the mappings A, S and T, we have that

$$F^{i}_{Ax^*-Ax_1}(\varepsilon) \geq F^{f(i)}_{Sx^*-Tx_1}\left(\frac{\varepsilon}{q(i)}\right) = F^{f(i)}_{Ax^*-x_1}\left(\frac{\varepsilon}{q(i)}\right) \geq$$

$$\geq \ldots \geq F^{f^n(i)}_{Ax^*-Ax_1}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \geq F^{g(i)}_{Ax^*-Ax_1}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \to 1$$

for $n \to \infty$ and $\varepsilon(i) = \dfrac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))}$ and for all $i \in I$, which means that

$$F^{i}_{Ax^*-x_1}(\varepsilon) = F^{i}_{Ax^*-Ax_1}(\varepsilon) = 1$$

for all $\varepsilon > 0$ and for $i \in I$. From the last equality we have that

$$Ax^* = x_1 \text{ ,}$$

which means that Ax^* is the unique fixed point for the mappings A, S and T.

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REZIME

TEOREME O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI FAMILIJE PRESLIKAVANJA U VEROVATNOSNIM LOKALNO KONVEKSNIM PROSTORIMA

$(X,\{F_i\}_{i\in I},t)$ je sekvencijalno kompletan verovatnosno lokalno konveksni prostor sa neprekidnom T-normom t. U radu je dokažana teorema u kojoj je dat potreban i dovoljan uslov za postojanje jedinstvene nepokretne tačke za dva preslikavanja S i T gde je $S : X \to X$, $T : X \to X$. Teorema glasi: Neprekidna preslikavanja S i T imaju jedinstvenu zajedničku nepokretnu tačku u X ako i samo ako postoji neprekidno preslikavanje $A : X \to SX \cap TX$ koje je komutativno sa S i T, AX je verovatnosno ograničen podskup od X i zadovoljeni su sledeći uslovi:

1. Za svako $i \in I$ postoji $q(i) > 0$ i $f(i) \in I$ tako da je

$$F^{i}_{Ax-Ay}(\varepsilon) \geq F^{f(i)}_{Tx-Sy}\left(\frac{\varepsilon}{q(i)}\right)$$

za svako $\varepsilon > 0$ i svako $x,y \in X$.

2. Za svako $i \in I$ postoje brojevi $n_i \in N$ i $Q(i) \in (0,1)$ tako da je

$$q(f^n(i)) \leq Q(i) \leq 1 \quad \text{za svako } n > n_i$$

3. Za svako $i \in I$ postoji $g(i) \in I$ tako da je

$$F_x^{f^n(i)}(\varepsilon) > F_x^{g(i)}(\varepsilon)$$

za svako $\varepsilon > 0$, svako $x \in X$ i svako $n \in N$.

Tada postoji jedan i samo jedan elemenat $x^* \in X$ takav da je Ax^* jedinstvena nepokretna tačka za preslikavanja A, S i T.