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ON THE PREDICTION OF A FUNCTIONAL OF GAUSSIAN RANDOM PROCESS

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Let $\{\xi(t),\ t\in T\}$ be a real Gaussian process (centered at the expectation: $E\xi(t)=0$) and let η be a integrable functional measurable with respect to σ-field $F(T)$ generated by $\{\xi(t),\ t\in T\}$. It is well-known that the conditional expectation $\hat{\xi}(t)=E(\xi(t)|F(S)|$, $S\subset T$, coincides with the projection of $\xi(t)$ on the Hilbert space $H^1(S)$ spanned on $\{\xi(t),\ t\in S\}$.

THEOREM. *The functional* $\hat{\eta}=E(\eta|F(S))$ *is measurable with respect to* σ*-field* $F_{\hat{\xi}}$ *generated by* $\{\hat{\xi}(t),\ t\in T\}$.

P r o o f. The conditional distribution (given $F(S)$) of η is determined by the family of the conditional distributions of the vectors $(\xi(t_1),\dots,\xi(t_n))$, $t_1,\dots,t_n\in T$. The conditional distribution of $(\xi(t_1),\dots,\xi(t_n))$ is Gaussian, so it is determined by the mean vector $(\xi(t_1),\dots,\xi(t_n))$ and the covariance matrix

$$B=\|b(t_i,t_j)\|,\quad i,j=1,\dots,n,$$

$$b(t_i,t_j)=E((\xi(t_i)-\hat{\xi}(t_i))(\xi(t_j)-\hat{\xi}(t_j))|F(S)).$$

But $\xi(t)-\hat{\xi}(t)$ is independent (for Gaussian process) of $F(S)$. So $b(t_i,t_j)$ is the constant $E(\xi(t_i)-\hat{\xi}(t_i))(\xi(t_j)-\hat{\xi}(t_j))$. In this way the conditional distribution of η depends only of $\hat{\xi}(t)$, $t\in T$, and its conditional expectation $\hat{\eta}$ depends only of $\hat{\xi}(t)$, $t\in T$.

Theorem is closely related to |2|, pp.73-78, but puts in evidence the evaluation of $\hat{\eta}$ by $\{\hat{\xi}(t)\}$.

Example 1. Let $H^n(S)$ be the linear closure of all polynomials of the variables $\xi(t)$, $t \in S$, the degree not greated than n. It is shown in |2| that $\eta \in H^n(T)$ implys $\hat{\eta} \in H^n(S)$. We precise this result showing that $\hat{\eta}$ belongs to the linear closure of all polynomias of the variables $\hat{\xi}(t)$, $t \in T$, the degree not greated than n. For this it is enough to see that

$E(\xi(t_1)\ldots\xi(t_n)|F(S))$ is a polynomials of $\hat{\xi}(t_1),\ldots \hat{\xi}(t_n)$ of the degree n:

$$E(\xi(t_1)\ \ldots\ \xi(t_n)|F(S)) =$$

$$= \frac{1}{(2\pi)^{n/2}(\det B)^{1/2}} \int\ldots\int x_1\ldots x_n \exp\{-\frac{1}{2}\sum_{i,j} B_{ij}(x_i - \hat{\xi}(t_i))(x_j-\hat{\xi}(t_j))\}dx_1\ldots\, dx_n = \frac{1}{(2\pi)^{n/2}(\det B)^{1/2}} \int\ldots$$

$$\ldots\int\prod_{k=1}^{n} (u_k+\hat{\xi}(t_k))\exp\{-\frac{1}{2}\sum_{i,j} B_{ij}u_iu_j\}du_1\ldots du_n =$$

$$= P_n(\hat{\xi}(t_1),\ldots\ \hat{\xi}(t_n))\,. \quad (B_{ij} \text{ is the cofactor of } b(t_i,t_j))\ .$$

Observe that

$$\frac{\partial}{\partial x_k} P_n(x_1,\ldots,x_k,\ldots,x_n) = P_{n-1}(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)\,.$$

APPLICATION TO THE PREDICTION

Let $T = [t_o,\infty)$ $S = [t_o,s]$, $t_o < s$. Then for fixed $t, t > s$, $\hat{\xi}(s,t) = E(\xi(t)|F_s)$, (we put $F(S) = F_s$), is the best (in the sence of the minimal variance) prediction of $\xi(t)$ by $\{\xi(u),\ u \in [t_o,s]\}$ $\hat{\xi}(s,t)$ coincides, for Gaussian process, with the best linear prediction which is widely elaborated for stationary process $\{\xi(t), -\infty < t < \infty\}$. In the terms of predication problem Theorem says that the prediction $\hat{\eta}$ of an integrable functional η of $\{\xi(u), u \in [s,\infty)\}$ by $\{\xi(u), u \in [t_o,s]\}$ is the functional of $\{\hat{\xi}(s,u), u \in [s,\infty]\}$

Example 2. Prediction of the time over a level by Gaussian process. Supposing the continuity of $\{\xi(t), t \geq 0\}$ the functional $\eta = \int_s^t I(\xi(u) > c)du$, ($I(.)$ is the indicator function), is the time over the level c by the process $\{\xi(u)\}$ during the time $[s,t]$. The prediction of η by $\{\xi(u), u \leq s\}$ is

$$\hat{\eta} = E(\eta|F_s) = \int_s^t P(\xi(u) > c|F_s)du = \int_s^t [1-\Phi(\frac{c-\hat{\xi}(s,u)}{\sqrt{b(u,u)}})]du,$$

where $\Phi(.)$ is the distribution function of a standard Gaussian variable.

Example 3. The problem of the prediction of the process $\{\zeta(t), t \geq o\}$, $\zeta(t) = f(\xi(t))$, where $f(.)$ is a non-random function, is posed in |1|. In the case $f(x) = x^2$ and the differentiable process $\{\xi(t), t \geq 0\}$ for which

$$\hat{\xi}(s,t) = \sum_{j=o}^{N-1} a_j(s,t)\xi^{(j)}(s) ,$$

(such process belongs to so called N-tiple Markov processes), the explicit formula for $\hat{\zeta}(s,t)$ in the terms of $\zeta(s),\dots \zeta^{(N-1)}(s)$ is given.

But generaly, because the conditional distribution of $\xi(t)$ given F_s is Gaussian with the parameters $\hat{\xi}(s,t)$ and $b(t,t)$, we have simple

$$\hat{\zeta}(s,t) = E(f(\xi(t))|F_s) = \frac{1}{\sqrt{2\pi b(t,t)}} \int f(x)\exp\{-\frac{1}{2}\frac{(x-\hat{\xi}(t,s))^2}{b(t,t)}\}dx =$$

$$= g(\hat{\xi}(t,s)) .$$

For instance, if $\{\xi(t)\}$ is as in |1| and $f^{-1}(.)$ is the differentiable function, we find $\hat{\zeta}(s,t)$ in terms of $\zeta(s),\dots\zeta^{(N-1)}(s)$.

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REZIME

O PREDVIDJANJU FUNKCIONALA GAUSOVOG SLUČAJNOG PROCESA

Pokazuje se da je predvidjanje funkcionala Gausovog procesa $\{\xi(t), t \in T\}$ funkcional od linearnog predvidjanja $\{\hat{\xi}(s,t), t \in T\}$.