Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu knjiga 11 (1981)
Review of Research Faculty of Science-University of Novi Sad, Volume 11(1981)

A UNIFORMLY CONVERGENT SCHEME WITH QUASI-CONSTANT FITTING FACTORS*

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1. INTRODUCTION

Consider the problem

$$\text{(BVP)}\qquad \begin{aligned} Lu(x) :&= -\varepsilon^2 u''(x) + g^2(x)u(x) = f(x), \quad x \in (0,1)\\ &u(o) = A, \quad u(1) = B. \end{aligned}$$

Here ε is a small positive constant, A and B are given constants, g and f are in $C^1[0,1]$, $g(o) = g(1)$ and $g(x) \geq \gamma$ on $[0,1]$ for some positive constant γ.

Under these assumptions (BVP) has a unique solution $u \in C^3[0,1]$ which in general displays a boundary layer at $x = o$ and $x = 1$ for "small" ε, |1|,|2|,|3|.

We want to solve the above problem by difference approximations on a non-uniform mesh which has more mesh points in the boundary regions than away from the boundary layers. In the case when the mesh is uniform our difference scheme is the same as in |2|.

Throughout the paper we shall let $C, C_1, \ldots$ denote positive constants that may take different values in different formulas, but that are always independent of ε and hk_i, $i=1,2,\ldots,n$. We assume that the parameter ε satisfies $o < \varepsilon \leq \varepsilon_o$ where ε_o is a positive constant.

2. DIFFERENCE APPROXIMATION FOR (BVP).

Consider a non uniform mesh

$$I_h = \{x_o=o,\ x_j=x_{j-1}+hk_k :\ j=1,2,\ldots,n\}$$

* Work on this paper was in part supported by the Self-Management Community of Interest for Scientific Research of Vojvodina.

with $n=2m$, $m \in N$, $k_j \in R$, $j=1,2,\dots,n$, $h^{-1}= \sum_{j=1}^{n} k_j$.

We want to choose $k_1,k_2,\dots,k_n$ in such a way that the following conditions are satisfied:

$$1 \le k_j \le k_{j+1} , \quad j=1,2,\dots,m-1$$
$$k_m = k_{m+1}$$
$$1 \le k_{j+1} \le k_j , \quad j=m+1, m+2,\dots,n-1 .$$

Let $\delta = g(o)/\varepsilon$ and for $i \in \{1,2,\dots,n-1\}$

$$(1) \qquad \alpha_i = \begin{cases} -k_i & \text{if } i \le m \\ k_{i+1} & \text{if } i > m \end{cases} , \quad \beta_i = \begin{cases} k_{i+1} & \text{if } i \le m \\ -k_i & \text{if } i > m \end{cases} .$$

Let u_i denote the approximate value (to be determined) for $u(x_i)$, $i=0,1,\dots,n$. We approximate (BVP) by

$$\text{(DBVP)} \qquad \begin{aligned} u_o &= A , \\ L^h u_i &:= L_o^h u_i + g^2(x_i)u_i = f(x_i), \quad i=1,2,\dots,n-1 \\ u_n &= B , \end{aligned}$$

where

$$L_o^h u_i := (da_i+(1-d)c_i)u_{i-1} + b_i u_i + ((1-d)a_i + dc_i)u_{i+1} ,$$

$$d = \begin{cases} 1 & \text{if } i \le m \\ 0 & \text{if } i > m \end{cases} ,$$

$$(2) \qquad a_i = -\frac{g^2(o)}{S}\sinh\delta\beta_i h, \quad c_i = \frac{g^2(o)}{S}\sinh\delta\alpha_i h , \qquad b_i = -a_i - c_i ,$$

$$(3) \qquad S = \sinh\delta(\beta_i - \alpha_i)h - \sinh\delta\beta_i h + \sinh\delta\alpha_i h .$$

In the Appendix it is proved that $a_i < 0$, $c_i < 0$, $b_i > 0$.

The above difference approximation for (BVP) is derived as follows. For a fixed $i \in \{1,2,\dots,n-1\}$ consider a linear system

(4) $\quad L_o^h F_j(x_i) = -\varepsilon^2 F_j''(x_i), \quad j=1,2,3$

where $F_1(x)=1$, $F_2(x)=\exp(\delta x)$, $F_3(x)=\exp(-\delta x)$. The solution of this system in given by (2) and (3).

THEOREM. *Suppose* $g, f \in C^1[0,1]$, $g(o)=g(1)$ *and* $g(x) \geq \gamma > o$ *on* $[0,1]$. *Then the solution* u *of (BVP) and the solution* u_i *of (DBVP) satisfy*

(5) $\quad |u(x_i) - u_i| \leq Chk_m, \quad i=0,1,\ldots,n$

where C *is indepedent of* i, h, k_m *and* ε.

P r o o f: We use in part: the notation, technique and some results from |2|. Since $g(o)=g(1)$ the solution of (BVP) can be written as

$$u(x) = v(x) + w(x) + z(x)$$

where

$$u_o(x) = f(x)/g^2(x), \quad v(x) = p\exp(-\delta x), \; w(x) = q\exp(\delta x),$$

$$p = (A-u_o(o) + (u_o(1)-B)\exp(-\delta))(1-\exp(-2\delta))^{-1}$$

$$q = (B-u_o(1) + (u_o(o)-A)\exp(-\delta))(1-\exp(-2\delta))^{-1}\exp(-\delta),$$

and $z(x)$ is the solution of

$$Lz(x) = f(x) - Lv(x) - Lw(x), \quad x \in (0,1)$$

$$z(o) = u_o(o), \quad z(1) = u_o(1).$$

For $z(x)$ we obtain, |2|,

$$|z^{(j)}(x)| \leq C(1+\varepsilon^{2-j}), \; j=0,1,2.$$

Defining the mesh functions $\{v_i\}$, $\{w_i\}$, $\{z_i\}$ by

$$L^h v_i = Lv(x_i), \quad v_o = v(o), \quad v_n = v(1),$$

$$L^h w_i = Lw(x_i), \quad w_o = w(o), \quad w_n = w(1),$$

$$L^h z_i = Lz(x_i), \quad z_o = z(o). \quad z_n = z(1),$$

we have

$$u_i = v_i + w_i + z_i.$$

Using (4) and (DBVP) it is easy to show that $v_i = v(x_i)$ and $w_i = w(x_i)$ for all $o \leq i \leq n$. Now, using these facts, we obtain

$$(6)\qquad L^h(u(x_i)-u_i) = L^h(z(x_i)-z_i) = L^h z(x_i) - Lz(x_i) \ .$$

We can write the (DBVP) as

$$A_h u_h = f_h \ ,$$

where

$$A_h = \begin{bmatrix} 1 & & & & & & & \\ a_1 & b_1+g^2(x_1) & c_1 & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & a_m & b_m+g^2(x_m) & c_m & & & & \\ & & c_{m+1} & \ddots & b_{m+1}+g^2(x_{m+1}) & a_{m+1} & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & c_{n-1} & b_{n-1}+g^2(x_{n-1}) & a_{n-1} \\ & & & & & & 1 \end{bmatrix},$$

$$u_h = [u_o,u_1,\dots,u_{n-1},u_n]^T, \qquad f_h = [A,f(x_1),\dots,f(x_{n-1}),B]^T \ .$$

If we denote $A_h = [(A_h)_{ij}] \in R^{n+1,n+1}$ we have

$$|(A_h)_{ii}| = \begin{cases} 1 & \text{for } i=1 \text{ and } i=n+1 \\ b_{i-1}+g^2(x_{i-1}) & \text{for } i=2,3,\dots,n \end{cases}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{n+1} |(A_h)_{ij}| = \begin{cases} 0 & \text{for } i=1 \text{ and } i=n+1 \\ -a_{i-1}-c_{i-1} & \text{for } i=2,3,\dots,n \end{cases} .$$

Since

$$\min_i(|(A_h)_{ii}| - \sum_{\substack{j=1\\ j\neq i}}^{n+1} |(A_h)_{ij}|) = \min_i(b_i+a_i+c_i+g^2(x_i),1) =$$

$$= \min_i(g^2(x_i),1) \geq \min(\gamma^2,1) > 0 \ ,$$

we can now apply Theorem 1. from [4] and we conclude that A_h is non singular and

$$(7)\qquad ||A_h^{-1}||_\infty \leq \frac{1}{\min(\gamma^2,1)} \ .$$

Now from (6) we obtain

(8) $$\|y\|_\infty \le \|A_h^{-1}\|_\infty \|r\|_\infty ,$$

where $y = [y_o, y_1, \dots, y_n]^T$, $r = [r_o, r_1, \dots, r_n]^T$ and

$$y_i = u(x_i) - u_i , \quad r_i = L^h z(x_i) - Lz(x_i), \; i=0,1,\dots,n.$$

We want to prove that

(9) $$\|r\|_\infty \le C h k_m$$

and then (5) follows directly from (7) and (8).

From (BVP) and (DBVP) follows

$$r_i = \varepsilon^2 (z''(x_i) - \frac{\delta^2}{S} \sinh\delta\beta_i h (z(x_{i-1}) + (\rho + \frac{\alpha_i - \beta_i}{\beta_i}) z(x_i)$$

$$- (\rho + \frac{\alpha_i}{\beta_i}) z(x_{i+1})))$$

where

$$= \frac{\sinh\delta\alpha_i h}{\sinh\delta\beta_i h} - \frac{\alpha_i}{\beta_i} .$$

Since

$$z(x_{i-1}) + (\rho + \frac{\alpha_i - \beta_i}{\beta_i}) z(x_i) - (\rho + \frac{\alpha_i}{\beta_i}) z(x_{i+1}) =$$

$$= \frac{\alpha_i(\alpha_i - \beta_i)}{2} \left(\frac{2z(x_{i-1})}{\alpha_i(\alpha_i - \beta_i)} + \frac{2z(x_i)}{\alpha_i \beta_i} + \frac{2z(x_{i+1})}{\beta_i(\beta_i - \alpha_i)} \right) +$$

$$+ \rho (z(x_i) - z(x_{i+1})) ,$$

and (see the Appendix)

(10) $$\frac{2z(x_{i-1})}{\alpha_i(\alpha_i - \beta_i)} + \frac{2}{\alpha_i \beta_i} z(x_i) + \frac{2z(x_{i+1})}{\beta_i(\beta_i - \alpha_i)} = h^2 z''(\tau_i) ,$$

$$\tau_i \in (x_{i-1}, x_{i+1}) ,$$

$$z''(x_i) = z''(\tau_i) + (x_i - \tau_i) z'''(\eta_i) , \quad \eta_i \in (\min(x_i,\tau_i), \max(x_i,\tau_i)),$$

$$z(x_i) - z(x_{i+1}) = -z'(\theta_i)(x_{i+1} - x_i) = -z'(\theta_i)\, hk_{i+1} ,$$

$$\theta_i \in (x_i, x_{i+1})$$

we have

$$r_i = \varepsilon^2(z''(x_i) - \frac{\delta^2}{S}\sinh\delta\beta_i h(\frac{\alpha_i(\alpha_i-\beta_i)}{2}h^2 z''(\tau_i) - \rho z''(\theta_i)hk_{i+1})) ,$$

and $r_i = \varepsilon^2(z''(\tau_i)(1-P)+(x_i-\tau_i)z'''(\eta_i)+\rho Phk_{i+1}z'(\theta_i))$,

where

$$(11) \qquad P = \frac{\alpha_i(\alpha_i-\beta_i)\delta^2 h^2}{2S}\sinh\delta\beta_i h .$$

Now we see that ($\tau_i \in (x_{i-1},x_{i+1})$, $|x_i-\tau_i| \le hk_{i+1}$)

$$|r_i| \le \varepsilon^2 \sup_{x\in[0,1]} |z''(x)||1-P| + \varepsilon^2 \sup_{x\in[0,1]} |z'''(x)|hk_{i+1}$$

$$+ \varepsilon^2 \sup_{x\in[0,1]} |z'(x)|\frac{\delta^2}{\beta_i S}(\beta_i\sinh\delta\alpha_i h - \alpha_i\sinh\delta\beta_i h)kh_{i+1} ,$$

and since (see the Appendix) $0 \le \beta_i\sinh\delta\alpha_i h - \alpha_i\sinh\delta\beta_i h \le \beta_i S$, $\frac{(\delta\beta_i h)^2}{4\sinh^2\frac{\delta\beta_i h}{2}} \le P \le 1 + \frac{(\delta\beta_i h)^2}{4}$, we obtain ($\max(k_i: i=1,2,\dots,n) = k_m$)

$$|r_i| \le \varepsilon^2 C_1|1-P| + \varepsilon^2 C_2(1+1/\varepsilon)hk_{i+1} + \varepsilon^2\delta^2 C_3(1+\varepsilon)hk_{i+1} ;$$

$$|r_i| \le 0.25\varepsilon^2\delta^2 C_1(hk_{i+1})^2 + C_4hk_{i+1} + C_5hk_{i+1} \le Chk_m ,$$

i.e. (9) is proved.

REMARK 1. In the case $k_i=1$, $i=1,2,\dots,n$ we have a uniform mesh and our scheme is the same as in [2]. In this case the estimate $|u(x_i)-u_i| \le Ch^2$, $i=0,1,\dots,n$ holds with the assumption that $g,f \in C^2[0,1]$, and then the proof of the theorem is not as complicated as in our case.

REMARK 2. From (5) it follows that the solution of (DBVP) converges to that of (BVP) as $\max_i(x_i-x_{i-1}) \to 0$, as $n \to \infty$ uniformly in ε.

3. APPENDIX

In this section we denote for a fixed $i \in \{1,2,...,n-1\}$ α_i by α and β_i by β.

LEMMA 1. *For* a_i, b_i, c_i *from (2), we have*

$$a_i < 0,\; c_i < 0,\; b_i > 0,\; i=1,2,...,n-1 .$$

P r o o f: We shall consider three cases: 1) $i = m$; 2) $i < m$; 3) $i > m$.

1) In this case $-\alpha = \beta = hk_m$ and

$$a_i = c_i = -0.5\, b_i = -(0.5\delta)^2 \sinh^{-2}(0.5\delta\beta h) < 0 .$$

2) In this case we have $1 \le -\alpha \le \beta$ and

$$S = S(\alpha,\beta) = \sinh\delta(\beta-\alpha) - \sinh\delta\beta h + \sinh\delta\alpha h$$

$$= \sum_{k=1}^{\infty} \frac{(\delta h)^{2k-1}}{(2k-1)!} \left((\beta-\alpha)^{2k-1} - \beta^{2k-1} + \alpha^{2k-1}\right)$$

$$= \sum_{k=2}^{\infty} \frac{(\delta h)^{2k-1}}{(2k-1)!} \sum_{j=1}^{2k-2} \binom{2k-1}{j} \beta^{2k-1-j} (-\alpha)^j > 0 ,$$

since $\delta h > 0$. Now we have

$$a_i = -\frac{g^2(o)}{S} \sinh\delta\beta h < 0, \quad (\beta > 0)$$

$$c_i = \frac{g^2(o)}{S} \sinh\delta\alpha h < 0, \quad (\alpha < 0)$$

$$b_i = -(a_i + c_i) > 0 .$$

3) In this case $1 \le -(-\alpha) \le -\beta$ and $S(\alpha,\beta) = -S(-\alpha,-\beta)$. Since $S(-\alpha,-\beta) > 0$, this is proved in 2), we obtain $S(\alpha,\beta) < 0$ and $a_i < 0$ $(\beta < 0)$, $c_i < 0$ $(\alpha > 0)$.

LEMMA 2. *If* $z \in C^2[0,1]$ *then (10) is satisfied.*

P r o o f: We have for $i \le m$

$$z(x_{i-1}) = z(x_i+\alpha h) = z(x_i) + \alpha h z'(x_i) + 0.5(\alpha h)^2 z''(\tau_1) ,$$
$$z(x_{i+1}) = z(x_i+\beta h) = z(x_i) + \beta h z'(x_i) + 0.5(\beta h)^2 z''(\tau_2) ,$$
$$\tau_1 \in (x_{i-1},x_i) , \quad \tau_2 \in (x_i,x_{i+1}) ,$$

and

$$\frac{2z(x_{i-1})}{\alpha(\alpha-\beta)} - \left(\frac{2}{\alpha(\alpha-\beta)} + \frac{2}{\beta(\beta-\alpha)}\right) z(x_i) + \frac{2z(x_{i+1})}{\beta(\beta-\alpha)} =$$

$$h^2\left(\frac{-\alpha}{\beta-\alpha} z''(\tau_1) + \frac{\beta}{\beta-\alpha} z''(\tau_2)\right) = h^2 z''(\tau) , \quad \tau \in (x_{i-1},x_{i+1}).$$

For $i > m$ the proof is the same.

LEMMA 3. *For* S *from (2) and* P *from (11) and for all* $1 \le i \le n-1$

(i) $0 \le \beta\sinh\delta\alpha h - \alpha\sinh\delta\beta h \le \beta S$,

(ii) $\dfrac{(\delta\beta h)^2}{4\sinh^2\frac{\delta\beta h}{2}} \le P \le 1 + \dfrac{(\delta\beta h)^2}{4}$,

(iii) $|1-P| \le \left(\dfrac{\delta\beta h}{2}\right)^2$.

P r o o f: First shall we consider the case $i \le m$.

(i): From $\beta \ge -\alpha \ge 1$ follows

$$-\alpha\beta(\beta^{2k-2} - \alpha^{2k-2}) \ge 0 , \quad \text{for } k \ge 1 ,$$

$$\sum_{k=1}^{\infty} \frac{(\delta h)^{2k-1}}{(2k-1)!} (\beta\alpha^{2k-1} - \alpha\beta^{2k-1}) \ge 0$$

and $\beta\sinh\delta\alpha h - \alpha\sinh\delta\beta h \ge 0$.

Now we have $(S\beta > 0)$ $\beta\sinh\delta\alpha h - \alpha\sinh\delta\beta h - \beta S =$

$$\sum_{k=2}^{\infty} \frac{(\delta h)^{2k-1}}{(2k-1)!} \left(-\alpha\beta(\beta^{2k-2} - \alpha^{2k-2}) - \beta \sum_{j=1}^{2k-2} \binom{2k-1}{j} \beta^{2k-1-j}(-\alpha)^j\right).$$

Since $-\alpha\beta(\beta^{2k-2}-\alpha^{2k-2})-\beta\beta^{2k-2}(-\alpha)(2k-1)-\beta\beta(-\alpha)^{2k-2}(2k-1) -$

$\beta \sum_{j=2}^{2k-3} \binom{2k-1}{j} \beta^{2k-1-j}(-\alpha)^j \le 0$, for $k \ge 2$, (ii) follows directly.

(ii) Using the elementary properties of hyperbolic functions we obtain

$$P = P(\alpha,\beta) = 0.5\alpha(\alpha-\beta)\delta^2h^2 \quad \frac{\cosh\frac{\delta\beta h}{2}}{\cosh\frac{\delta(\beta-2\alpha)h}{2} - \cosh\frac{\delta\beta h}{2}} .$$

Now we have

$$\frac{\partial P}{\partial\alpha}(\alpha,\beta) = 0.5\delta^2h^2\cosh\frac{\delta\beta h}{2} \quad Q/(\cosh\frac{\delta(\beta-2\alpha)h}{2} - \cosh\frac{\delta\beta h}{2})^2 ,$$

where

$$Q = -(\beta-2\alpha)(\cosh\frac{\delta(\beta-2\alpha)h}{2} - \cosh\frac{\delta\beta h}{2}) + \alpha(\alpha-\beta)\delta h\sinh\frac{\delta(\beta-2\alpha)h}{2},$$

and we see that $\frac{\partial P}{\partial\alpha}(\alpha,\beta) > 0$ if and only if $Q > 0$, for $\beta \geq -\alpha \geq 1$. We write Q in the form

$$Q = -(\beta-2\alpha)\sum_{k=o}^{\infty}\frac{(0.5\delta h)^{2k}}{2k!}((\beta-2\alpha)^{2k} - \beta^{2k}) +$$

$$+ \alpha(\alpha-\beta)\sum_{k=1}^{\infty}\frac{(0.5\delta h)^{2k-1}}{(2k-1)!}(\beta-2\alpha)^{2k-1} ,$$

$$Q = -(\beta-2\alpha)\sum_{k=1}^{\infty}\frac{(0.5\delta h)^2}{(2k)!}R_k .$$

where $\quad R_k = (\beta-2\alpha)^{2k}(1 - \frac{4k\alpha(\alpha-\beta)}{(\beta-2\alpha)^2}) - \beta^{2k}$.

We immediately see that $R_1 = 0$, $R_2 = -16\alpha^2(\alpha-\beta)^2 < 0$, and that $R_{k+1} = \beta^2R_k - 16\alpha^2k(\beta-2\alpha)^{2k-2}(\alpha-\beta)^2$, $k=1,2,\ldots$. Now, from $R_k < 0$ follows $R_{k+1} < 0$ and we conclude that $\frac{\partial P}{\partial\alpha}(\alpha,\beta) > 0$ for $\beta \geq -\alpha \geq 1$. From this we obtain $(\alpha \geq -\beta)$

$$P(\alpha,\beta) \geq P(-\beta,\beta) = \frac{(\delta\beta h)^2}{2}\sinh^{-2}(\frac{\delta\beta h}{2}),$$

which is the first part of (ii).

Now we want to prove $P(\alpha,\beta) \leq 1 + (0.5\delta\beta h)^2$. This inequality is equivalent to $T \leq 0$, where

$$T = 0.5\alpha(\alpha-\beta)\cosh(0.5\delta\beta h) - (1+(0.5\delta\beta h)^2(\cosh(0.5\delta(\beta-2\alpha)h) -$$
$$- \cosh(0.5\delta\beta h)) .$$

Using Taylor expansions we obtain

$$T = 0.5\alpha(\alpha-\beta)\delta^2h^2 + \alpha(\alpha-\beta)\delta^2h^2(-8-\delta^2\beta^2h^2)/16 + \sum_{k=2}^{\infty} \frac{(0.5\delta h)^2 L_k}{(2k)!},$$

where for $k \geq 2$

$$L_k = (1+(0.5\delta\beta h)^2)(\beta^{2k} - (\beta-2\alpha)^{2k}) + \beta^{2k}\, 0.5\alpha(\alpha-\beta)\delta^2h^2 .$$

Since $L_2 = \alpha(\alpha-\beta)(-1.5\delta^2h^2\beta^4 - 8\beta^2 - 4(4+\delta^2\beta^2h^2)\alpha(\alpha-\beta) < 0$

and $\quad L_k = \beta^2 L_k - \alpha(\alpha-\beta)(4+(\delta\beta h)^2)(\beta-2\alpha)^{2k}$

we conclude that $L_k < 0$, $k=2,3,\ldots$ and that $T \leq 0$.

(iii) Introducing the notation $t = 0.5\delta\beta h$, $y(t) = t^2\sinh^{-2}t$ from (ii) we have $y(t) \leq P \leq 1 + t^2$. Since $y(t) \leq 1$ and $1-y(t) \leq t^2$ we obtain

$$|1-P| \leq \max\,(t^2,\ 1-y(t)) = t^2 = (0.5\delta\beta h)^2 .$$

In the case $i > m$ we have $-\beta \geq -(-\alpha) \geq 1$, $S < 0$, $S\beta > 0$, $S(\alpha,\beta) = -S(-\alpha,-\beta)$, $P(-\alpha,-\beta) = P(\alpha,\beta)$ so that we obtain (i), (ii) and (iii) as in the case $i \leq m$.

REFERENCES

|1| Bakhvalov, N.S., *The optimization of methods of solving boundary value problems with a boundary layer, USSR Computational Math. and Math.Phys., Vol. 9, No 4, pp. 139-166 (1969).*

|2| Doolan, E.P., J.J.H.Miller, W.H.A. Schilders, *Uniform numerical methods for problems with initial and boundary layers, Boole Press, Dublin, 1980.*

|3| Pearson, C.E., *On a differential equation of boundary layer type, J.Math. Phys., 47, pp.134-154 (1968).*

|4| Varah, J.M., *A lower bound for the smallest singular value of a matrix, Linear Algebra Appl. 11, pp. 3-5 (1975).*

REZIME

UNIFORMNO KONVERGENTNA ŠEMA SA KVAZIKONSTANTNIM FITING FAKTORIMA

U radu se posmatra problem (BVP) pod pretpostavkama : $f,g \in C^1[0,1]$, $g(x) \geq \gamma > 0$, $x \in [0,1]$, $\varepsilon_o \geq \varepsilon > 0$, $\varepsilon_o, \varepsilon, \gamma, A, B \in R$. Jedinstveno rešenje $u \in C^3[0,1]$ ima u opštem slučaju fenomen graničnog sloja u x=0 i x=1. Da bi što više tačaka bilo u graničnim slojevima za fiksan ukupan broj tačaka mreže diskretizacije, koristi se neekvidistantna mreža. Konvergencija uniformna po ε dobijenog diferencnog postupka dokazana je u teoremi pod istim pretpostavkama kao odgovarajuća teorema za ekvidistantnu mrežu iz [2].