Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 11 (1981)
Review of Research Faculty of Science-University of Novi Sad, Volume 11(1981)

SOME FINITE-DIFFERENCE SCHEMES FOR A SINGULAR PERTURBATION PROBLEM ON A NON-UNIFORM MESH*

Dragoslav Herceg and Relja Vulanović

Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

1. INTRODUCTION

In paper [1] Bahvalov suggested a finite-difference method on a non-uniform mesh for solving boundary value problem (1). The mesh points in that method are completely determined as values of function $\lambda(s)$ at equidistant points, in such a way that the consistency error is uniform in perturbation parametat ε. A second order convergence uniform in ε is achived.

In the paper presented here we shall consider meshes formed with more freedom in choosing mesh pointes, yet, implying convergence uniform in ε. This enables us to achieve a greater number of mesh points in narrow regions of boundary layers.

Consider the problem

$$(1)\qquad \begin{aligned} &Lx(t) := -\varepsilon^2 x''(t) + b^2(t)x(t) = f(t), \quad t \in (0,1) \\ &x(o) = X_o, \quad x(1) = X_1 \\ &b(t) \geq \beta > o, \quad t \in [0,1] \end{aligned}$$

where $o < \varepsilon \leq \varepsilon_o, \varepsilon, \varepsilon_o, X_o, X_1, \beta \in \mathbb{R}$. If $b, f \in C[0,1]$, then there exists a unique solution $x \in C^2[0,1]$ to problem (1), [7].

Further on, C will denote each positive constant independent of ε.

For problem (1) we can state ([1], [4], [8]):

LEMMA. *Let* L *be the operator in (1) and* y *any smooth function such that* $|(Ly(t))^{(j)}| \leq C$, $j=0,1,\ldots,j_o$, $t \in [0,1]$.

* Work on this research study was in part supported by the Self-Management Community of Interest for Scientific Research of Vojvodina.

Then for $j=0,1,\dots,j_o+2$ *and* $t \in [0,1]$

(2) $$|y^{(j)}(t)| \le C(1+\varepsilon^{-j}\exp(-tb(0)/\varepsilon) + \varepsilon^{-j}\exp(-(1-t)b(1)/\varepsilon)).$$

If, moreover, $b^2(0)y(0)=Ly(0)$ and $b^2(1)y(1)=Ly(1)$, then

(2a) $$|y^{(j)}(t)| \le C(1+\varepsilon^{2-j}\exp(-tb(0)/\varepsilon) + \varepsilon^{2-j}\exp(-(1-t)b(1)/\varepsilon)),$$
$$j=0,1,\dots,j_o+2 \quad t \in [0,1].$$

In this paper we shall use Bahvalov's function $\lambda(s)$, |1|. Here we shall give its definition and some properties.

Let $a > o$ and $q \in (o,o.5)$ be arbitrary constants independent of ε and

$$\psi(s) = a\varepsilon \ln(q/(q-s)), \quad s \in [o,q).$$

Let $(\alpha,\psi(\alpha))$ denote the point of contact of a tangent line taking the value 0.5 at 0.5, to the curve $\psi(s)$. α is the solution of equation

$$\psi(x) = 0.5 + \psi'(x)(x-0.5) .$$

It is easy to prove that this equation has a unique solution $\alpha \in [0,q)$ when $a\varepsilon \le q < 0.5$. We can determine α from $\alpha = q - q/x_\infty$ where x_∞ is a limit of the series

$$x_o = 1, \quad x_{k+1} = x_k - h(x_k)/h'(x_k), \quad k=0,1,\dots$$

and $$h(x) = \ln x - (1-0.5/q)x + 1-0.5/(a\varepsilon) .$$

Now we can define $\lambda(s)$. For $a\varepsilon > q$ we take $\lambda(s) = s$, $s \in [0,1]$ and for $a\varepsilon \le q$ let

$$\lambda(s) = \begin{cases} \psi(s) & , \ s \in [0,\alpha] \\ \psi(\alpha) + \psi'(\alpha)(s-\alpha) & , \ s \in [\alpha,0.5] \\ 1 - \lambda(1-s) & , \ s \in [0.5,1] \end{cases} .$$

For $\lambda(s)$ we have $\lambda(s) \ge 0$, $\lambda'(s) > 0$, $s \in [0,1]$ and when $a\varepsilon \le q$

(3) $$\lambda''(s) \begin{cases} \ge 0 & , \ s \in [0,\alpha] \\ = 0 & , \ s \in (\alpha, 1-\alpha). \\ \le 0 & , \ s \in [1-\alpha, 1] \end{cases}$$

The contact point of the tangent, parallel with line $(s-q)/(1-2q)$, to the curve $\psi(s)$ is denoted by $(\alpha_1,\psi(\alpha_1))$. We have $\alpha \leq \alpha_1$ and $\alpha_1 = q - q_1\varepsilon$, $q_1 = a(1-2q)$.

By $(\alpha_2,\psi(\alpha_2))$ we denote the contact point of the tangent, parallel with line s, to the curve $\psi(s)$. Now $\alpha_2 \leq \alpha$, $\alpha_2 = q - a\varepsilon$.

2. DISCRETISATION MESH

We form the discretisation of (1) on the mesh $I_h = \{t_o, t_1,\ldots,t_n\}$, $n = 2m$, $m \geq 2$, $m \in N$, where

$$t_o = 0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = 0.5 ,$$

(4)

$$t_{n-i} = 1 - t_i, \quad i = 0,1,\ldots,m-1 .$$

Since $t_o = 0$, $t_m = 0.5$ and the points t_i for $i = m+1,\ldots,n$ are determined when t_i, $i = 1,2,\ldots,m-1$, are known, we shall, when constructing the mesh I_h, give only the points t_i, $i=1,2,\ldots,m-1$. We shall consider two cases: case A and case B of mesh contruction.

For $a\varepsilon > q$ each non-uniform mesh I_h is convenient for the numerical solution of problem (1) when discretisation is given as in |1|, because we achieve convergence uniform in ε. From now on we shall consider only the case $a\varepsilon \leq q$, i.e. $\varepsilon_o = q/a$.

We shall not consider the case $b^2(0)y(0) = Ly(0)$ and $b^2(1)y(1) = Ly(1)$ when the uniform convergence can be easily proved by using (2a).

Let $\tau(\varepsilon)$ be such a function that $\tau(\varepsilon) < 0.5$ and

$$\exp(-\tau(\varepsilon)/(a\varepsilon)) \leq C\,\varepsilon , \quad \varepsilon \in (0,\varepsilon_o], \tag{5}$$

for example $\tau(\varepsilon) = \varepsilon^{(k-1)/k}$ for $k \geq 2$, ε_o small enough, or $\tau(\varepsilon) = \lambda(\alpha_2)$. Define $K = s_\varepsilon/n_o$, where

$$s_\varepsilon = q(1-\exp(-\tau(\varepsilon)/(a\varepsilon))), \quad n_o \geq 2q/(1-2q), \quad n_o \in N .$$

Then $K(n_o+1) < 0.5$, $n_o + 2 \leq m$ and $s_\varepsilon < q$.

Case A.

Let $t_i = \lambda(Ki)$, $i=1,2,\ldots,n_o+1$. If $n_o+2 = m$, we have $t_{n_o+2} = 0.5$ and if $n_o+2 < m$, we take the points t_i, $i=n_o+2,\ldots,m-1$, arbitrarily, but with property (4).

Then, using (3), we have

$$(6) \qquad \begin{aligned} t_{i+1}-t_i &\geq K\,\lambda'(Ki) \geq t_i-t_{i-1} \ , \quad i=1,2,\ldots,n_o \ , \\ t_{i+1}-t_i &\leq K\,\lambda'(Ki) \leq t_i-t_{i-1} \ , \quad i=n-n_o,\ldots,n-1 \ . \end{aligned}$$

Case B.

For $i=1,2,\ldots,n_o+1$ we choose t_i as in case A, and for $i=n_o+2,\ldots,m$ let it be possible to choose t_i so that

$$\begin{aligned} & t_{n_o+2}-t_{n_o+1} = t_{n_o+1}-t_{n_o} \ , \\ & t_i-t_{i-1} \leq t_{i+1}-t_i \leq t_i-t_{i-p}, \quad i=n_o+2,\ldots,m-1 \ , \end{aligned}$$

for some $p = p(i) \in \{2,3,\ldots,i-n_o\}$.

3. DISCRETISATION OF (1)

We shall have two discretisations of problem (1) for both cases, A and B, of mesh construction.

Case A.

Define

$$L_h^1x(t_i) := a_i^1x(t_{i-1}) + b_i^1x(t_i) + c_i^1x(t_{i+1}) \ , \quad i=1,2,\ldots,n-1$$

where

$$a_i^1 = \frac{-2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})} < 0 \ ,$$

$$b_i^1 = \frac{2}{(t_i-t_{i-1})(t_{i+1}-t_i)} > 0 \ ,$$

$$c_i^1 = \frac{-2}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} < 0 \ .$$

If $b,f \in C^1[0,1]$, then

(7) $\rho_i^1 := -x''(t_i) - L_h^1 x(t_i) = (\theta_i^1 - t_i)x'''(\theta_i^2)$,

with $t_{i-1} \le \theta_i^k \le t_{i+1}$, $k=1,2$, $i=1,2,\dots,n-1$, and if $b,f \in C^2[0,1]$ then

(8) $$\rho_i^1 = \frac{1}{3}(t_{i+1}-2t_i+t_{i-1})x'''(t_i) + \frac{(t_{i+1}-t_i)^3}{12(t_{i+1}-t_{i-1})}x^{iv}(\sigma_i^2) + \frac{(t_i-t_{i-1})^3}{12(t_{i+1}-t_{i-1})}x^{iv}(\sigma_i^1),$$

$t_{i-1} \le \sigma_i^1 \le t_i \le \sigma_i^2 \le t_{i+1}$, $i=1,2,\dots,n-1$.

The system of linear equations

$$x_h(t_o) = X_o, \quad x_h(t_n) = X_1$$

(9) $$\varepsilon^2 L_h^1 x_h(t_i) + b(t_i)x_h(t_i) = f(t_i), \quad i=1,2,\dots,n-1,$$

with solution $x_h = (x_h(t_o), x_h(t_1), \dots, x_h(t_n))^T \in \mathbb{R}^{n+1}$ is a discretisation of (1).

We can write discretisation (9) in the form:

$$\varepsilon^2 A_h x_h + B_h x_h = f_h,$$

where $B_h = \operatorname{diag}(0, b(t_1), \dots, b(t_{n-1}), 0) \in \mathbb{R}^{n+1,n+1}$,

$f_h = (X_o, f(t_1), \dots, f(t_{n-1}), X_1)^T \in \mathbb{R}^{n+1}$,

$$A_h = \begin{bmatrix} \varepsilon^{-2} & & & & & \\ a_1^1 & b_1^1 & c_1^1 & & & \\ & a_2^1 & b_2^1 & c_2^1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1}^1 & b_{n-1}^1 & c_{n-1}^1 \\ & & & & & \varepsilon^{-2} \end{bmatrix}$$

Elements of matrix A_h equal to zero are not indicated.

Case B.

Now we shall define the discretisation of (1) on the mesh for case B. For $i=0,1,\dots,n_o$ and $i=n-n_o,\dots,n$ the discretisation is given as in (9), and otherwise by

$$\varepsilon^2 L_h^2 x_h(t_i) + b(t_i)x_h(t_i) = f(t_i), \text{ for } i=n_o+1,\dots,m ,$$

$$\varepsilon^2 L_h^3 x_h(t_i) + b(t_i)x_h(t_i) = f(t_i), \text{ for } i=m+1,\dots,n-n_o-1,$$

where

$$L_h^2 x(t_i) = d_i x(t_{i-p}) + c_i x(t_{i-1}) + b_i x(t_i) + a_i x(t_{i+1}),$$

$$L_h^3 x(t_i) = a_i x(t_{i-1}) + b_i x(t_i) + c_i x(t_{i+1}) + d_i x(t_{i+p}).$$

Natural numbers $p=p(i)$ for $i=n_o+2,\dots,m-1$ are given in the construction of the mesh for case B, and for $i=m+1,\dots,n-n_o-2$ we take $p(i)=p(n-i)$. Since $t_{n_o+2}-t_{n_o+1} = t_{n_o+1}-t_{n_o}$ and $t_{m+1}-t_m=t_m-t_{m-1}$ imply $d_{n_o+1}=d_m=0$, we can formally take $p(n_o+1) = p(m) = 0$ and $p(n-n_o-1)=0$.

The coefficients in the given schemes are, |5|:

$$a_i = \frac{2(z_2+z_3)}{z_1(z_1-z_2)(z_1-z_3)}, \qquad b_i = \frac{-2(z_1+z_2+z_3)}{z_1 z_2 z_3},$$

$$c_i = \frac{2(z_1+z_3)}{z_2(z_2-z_1)(z_2-z_3)}, \qquad d_i = \frac{2(z_1+z_2)}{z_3(z_3-z_1)(z_3-z_2)}$$

where $z_k = z_k(i)$, $k=1,2,3$, :

$$z_1 = \begin{cases} t_{i+1}-t_i, & \text{for } L_h^2 \\ -(t_i-t_{i-1}), & \text{for } L_h^3 \end{cases} \qquad z_2 = \begin{cases} -(t_i-t_{i-1}), & \text{for } L_h^2 \\ t_{i+1}-t_i, & \text{for } L_h^3 \end{cases}$$

$$z_3 = \begin{cases} -(t_i-t_{i-p(i)}), & \text{for } L_h^2 \\ t_{i+p(i)}-t_i, & \text{for } L_h^3 \end{cases}$$

For a_i, b_i, c_i, d_i from L_h^2 we can state $a_i<0$, $b_i>0$, $c_i\le 0$, $d_i\le 0$.

For k=2,3 and $b,f \in C^2[0,1]$ we have

$$(10) \qquad \rho_i^k := -x''(t_i) - L_h^k x(t_i) = \frac{1}{12}(z_1z_2+z_1z_3+z_2z_3)x^{iv}(\delta_i^k) ,$$

$$t_{i-p} \le \delta_i^2 \le t_{i+1} , \quad t_{i-1} \le \delta_i^3 \le t_{i+p}$$

As in case A, we can now write the discretisation in the form

$$\varepsilon^2 A_h x_h + B_h x_h = f_h$$

where B_h and f_h are defined as before. The matrix A_h for this case is formed analogously as in case A.

4. CONSISTENCY ERROR

Define $r = (0,r_1,r_2,\dots,r_{n-1},0)^T \in \mathbb{R}^{n+1}$ with $r_i = \varepsilon^2 \rho_i^1$, $i=1,2,\dots,n-1$, where ρ_i^1 is given by (7) or (8), and let R denote:

$$R = (0,R_1,R_2,\dots,R_{n-1},0)^T \in \mathbb{R}^{n+1} ,$$

$$R_i = \varepsilon^2 \begin{cases} \rho_i^1 & , i=1,2,\dots,n_o \\ \rho_i^2 & , i=n_o+1,\dots,m \\ \rho_i^3 & , i=m+1,\dots,n-n_o-1 , \\ \rho_i^1 & , i=n-n_o,\dots,n \end{cases}$$

where ρ_i^k , k=2,3, are given by (10). For $x=(x_1,x_2,\dots,x_{n+1})^T \in \mathbb{R}^{n+1}$ let $\|x\|_\infty = \max_{1\le i\le n+1} |x_i|$, and let h_A , h_B denote:

$$h_A = \max_{n_o+1\le i\le m} (t_i - t_{i-1}) , \qquad h_B = \max_{n_o+2\le i\le m-1} (t_i - t_{i-p(i)})$$

and $d = \min(b(0), b(1))$.

Case A.

THEOREM 1. *Let* $b,f \in C^1[0,1]$ *and* $a \geq 1/d$. *Then*

$$\|r\|_\infty \leq C(1/n_o + h_A) \ ,$$

where r *is given by* ρ_i^1 *from (7).*

P r o o f: Using (7) and (2) we have

$$|r_i| \leq \varepsilon^2 \max(t_{i+1}-t_i,\ t_i-t_{i-1})C(1+\varepsilon^{-3}(\exp(-t_{i-1}b(0)/\varepsilon) +$$

$$+ \exp(-(1-t_{i+1})b(1)/\varepsilon))),\quad i=1,2,\dots,n-1 \ .$$

If $n_o+1 \leq i \leq n-n_o-1$, then

$$t_{i-1} \geq t_{n_o} = \lambda(s) = \tau(\varepsilon) \quad \text{and}$$
$$t_{i+1} \leq t_{n-n_o} = 1-t_{n_o} = 1-\tau(\varepsilon)$$

and because of (5) and $ad \geq 1$, we conclude

$$(11)\qquad |r_i| \leq C\max(t_{i+1}-t_i,\ t_i-t_{i-1}) \ .$$

When $1 \leq i < n_o+1$, we have $t_{i+1} < 0.5$, and $\varepsilon^{-1}\exp(-(1-t_{i+1})b(1)/\varepsilon)$ is bounded uniformly in ε. From this and because of (6) we get

$$|r_i| \leq C(t_{i+1}-t_i)(1+\varepsilon^{-1}\exp(-t_{i-1}b(0)/\varepsilon)) \leq$$
$$\leq C K \lambda'(K(i+1))(1+\varepsilon^{-1}\exp(-t_{i-1}b(0)/\varepsilon)) \leq$$
$$\leq \frac{C}{n_o}\lambda'(K(i+1))(1+\varepsilon^{-1}\exp(-t_{i-1}b(0)/\varepsilon)) \ .$$

Now let us show

$$(12)\qquad |r_i| \leq \frac{C}{n_o} \ .$$

Since

$$(13)\qquad \lambda'(s) \leq \psi'(\alpha) \leq \psi'(\alpha_1) = 1/(1-2q) \ ,$$

we just have to prove

$$(14)\qquad \lambda'(K(i+1))\varepsilon^{-1}\exp(-t_{i-1}b(0)/\varepsilon) \leq C \ .$$

1. $\alpha_2 \le K(i-1)$. Then we have $\lambda(\alpha_2) \le t_{i-1}$, and

$$\exp(-t_{i-1}b(0)/\varepsilon) \le \exp(-\lambda(\alpha_2)b(0)/\varepsilon) =$$
$$= ((q-\alpha_2)/q)^{ab(0)} = (a\varepsilon/q)^{ab(0)} ,$$

from where, using (13), we get (14), because $a\varepsilon \le q$ and $ad \ge 1$.

2. $K(i-1) < \alpha_2$. Now we consider two cases.

2.1. $K(i-1) \le \min(\alpha, q-4K)$. We conclude

$$\exp(-t_{i-1}b(0)/\varepsilon) = [(q-K(i-1))/q]^{ab(0)} \quad \text{and}$$
$$\lambda'(K(i+1)) \le \psi'(K(i+1)) = a\varepsilon/(q-K(i+1)) \le 2a\varepsilon/(q-K(i-1)) ,$$

because from $K(i-1) \le q-4K$ we get $0.5(q-K(i-1)) \le q-K(i+1)$. The above relations imply (14).

2.2. $\min(\alpha, q-4K) < K(i-1) < \alpha_2$. Since $\alpha \ge \alpha_2$, we have

$$q-4K < K(i-1) < \alpha_2 = q-a\varepsilon ,$$

(15) $$\varepsilon < 4K/a < \frac{4q}{a}\frac{1}{n_o} .$$

Now consider the inequality

$$|r_i| \le 2\varepsilon^2 \max_{t\in[t_{i-1},t_{i+1}]} |x''(t)| \le C\varepsilon^2(1+\varepsilon^{-2}\exp(-t_{i-1}b(0)/\varepsilon)+$$
$$+ \varepsilon^{-2}\exp(-(1-t_{i+1})b(1)/\varepsilon)) \le C(\varepsilon^2+\exp(-t_{i-1}b(0)/\varepsilon)) .$$

Because of

$$\exp(-t_{i-1}b(0)/\varepsilon) < (4K/q)^{a\,b(0)} < (4/n_o)^{a\,b(0)}$$

and (15), we get (12) directly.

For the case $n-n_o-1 < i \le n-1$ we can prove (12) analogously .

From (11) and (12) we get the statement of the theorem.

THEOREM 2. *Let* $b, f \in C^2[0,1]$ *and* $ad \geq 2$. *Then*

$$\|r\|_\infty \leq C(1/n_o^2 + h_A) ,$$

where r *is given by* ρ_i^1 *from (8).*

P r o o f: When $n_o+1 \leq i \leq n_o-1$, we can prove relation (11) in the same way as in Theorem 1. For other values of i we shall prove

(16) $$|r_i| \leq C / n_o^2$$

Using (6) we have for $1 \leq i < n_o + 1$

$$|r_i| \leq \varepsilon^2 (P+Q),$$

$$P = \frac{1}{3} (t_{i+1}-2t_i+t_{i-1}) |x'''(t_i)|,$$

$$Q = \frac{1}{12} (t_{i+1}-t_i)^2 2 \cdot \max_{t\in[t_{i-1},t_{i+1}]} |x^{iv}(t)|.$$

We can show that $\varepsilon^2 Q \leq C/n_o^2$ analogously to Theorem 1.

Now let us consider P. We have

$$t_{i+1}-2t_i+t_{i-1} \leq K^2\lambda''(K(i+1)) \leq K^2 \psi''(\alpha_1) =$$

$$= K^2 a\varepsilon/(q-\alpha_1)^2 \leq C/(n_o^2 \varepsilon) \quad .$$

Using the above inequality we can prove that $\varepsilon^2 P \leq C/n_o^2$ when $Ki \geq \alpha_2$, in the same way as in part 1. of the proof of Theorem 1.

When $Ki < \alpha_2$ and $Ki \leq \min(\alpha, q-2K)$, we use

$$\lambda''(K(i+1)) = a\varepsilon/(q-K(i+1))^2 \leq 4a\varepsilon/(q-Ki)^2$$

to conclude the same fact, analogously to part 2.1. of the proof of Theorem 1.

When $q-2K < Ki < \alpha_2$, the proof is the same as in the previous theorem.

For $n-n_o-1 < i \leq n-1$ the proof is analogous.

Case B.

THEOREM 3. *Let* $b,f \in C^2[0,1]$ *and* $ad \geq 2$. *Then* $\|R\|_\infty \leq C(1/n_o^2 + h_B^2)$.

P r o o f: When $i=1,2,\dots,n_o$ and $i=n-n_o,\dots,n$, we have

$$|R_i| = |r_i| \leq C/n_o^2 .$$

The proof is the same as in the previous theorem.

For other i we have, because of (10),

$$|R_i| \leq C\varepsilon^2 \max(z_2^2, z_3^2)(1+\varepsilon^{-4}\exp(-t_{i-p(i)}b(0)/\varepsilon)),\ i=n_o+1,\dots,m$$

$$|R_i| \leq C\varepsilon^2 \max(z_2^2, z_3^2)(1+\varepsilon^{-4}\exp(-(1-t_{i+p(i)})b(1)/\varepsilon)),$$

$$i=m+1,\dots,n-n_o-1 .$$

But here we have $t_{i-p(i)} \geq \tau(\varepsilon)$ and $t_{i+p(i)} \leq 1-\tau(\varepsilon)$, and from (5) and $ad \geq 2$ we conclude

$$|R_i| \leq C \max_{n_o+2 \leq j \leq m-1} (t_j - t_{j-p(j)})^2, \quad n_o+1 \leq i \leq n-n_o-1.$$

This completes the proof of the theorem.

5. CONVERGENCE UNIFORM IN ε

Discretisations of (1) in both cases of paragraph 3. can be written in the form

$$(17) \qquad \varepsilon^2 A_h x_h + B_h x_h = f_h ,$$

where $A_h, B_h \in \mathbb{R}^{n+1,n+1}$, $x_h, f_h \in \mathbb{R}^{n+1}$ are defined as in paragraph 3. Let x^h denote the restriction of the exact solution of problem (1) to mesh I_h, and let $r, R \in \mathbb{R}^{n+1}$ be the same as in the previous paragraph.

Now we have

$$(18) \qquad \varepsilon^2 A_h x^h + B_h x^h = f_h - r$$

for case A and

(19) $$\varepsilon^2 A_h x^h + B_h x^h = f_h - R$$

for case B.

From (17) and (18) we get

(20) $$\varepsilon^2 A_h (x^h - x_h) + B_h (x^h - x_h) = - r$$

and, analogously, for case B:

(21) $$(\varepsilon^2 A_h + B_h)(x^h - x_h) = - R .$$

THEOREM 4. *The matrices* $\varepsilon^2 A_h + B_h$ *in both cases* A *and* B *are regular and*

$$\|(\varepsilon^2 A_h + B_h)^{-1}\|_\infty \le 1/\min(\beta^2, 1) .$$

P r o o f: For matrix $C_h = [c_{ij}] \in \mathbb{R}^{n+1,n+1}$ defined by

$$C_h = \varepsilon^2 A_h + B_h$$

we have

$$c_{ii} > 0, \quad i = 0, 1, \dots, n ,$$

$$c_{ij} \le 0, \quad i \ne j,\ i,j = 0,1,\dots,n .$$

Since $a_i^1 + b_i^1 + c_i^1 = 0$, $i=1,2,\dots,n$ and

$$a_i + b_i + c_i + d_i = 0, \quad i=1,2,\dots,n ,$$

we get

$$\sum_{\substack{j=0\\ j\ne i}}^{n} |c_{ij}| = - \sum_{\substack{j=0\\ j\ne i}}^{n} c_{ij} = \begin{cases} \bar{b}_i , & i=1,2,\dots,n-1 \\ 0 , & i=0,n \end{cases}$$

where $\bar{b}_i$ denotes b_i^1 or b_i. Now

$$S := \min_{0\le i\le n} \Big(|c_{ii}| - \sum_{\substack{j=0\\ j\ne i}}^{n} |c_{ij}|\Big) = \min_{1\le i\le n-1} (c_{ii} - \bar{b}_i, 1) .$$

Because of $c_{ii} = \bar{b}_i + b^2(t_i)$ and $b(t_i) \ge \beta$, $i=1,2,\dots,n-1$, we conclude

$$S \geq \min(\beta^2, 1) > 0 \ .$$

Then Theorem 1. from |9| implies that C_h is a regular matrix and

$$\|C_h^{-1}\|_\infty \leq 1/\min(\beta^2,1) \ ,$$

which completes the proof.

THEOREM 5. *Let* x_A^h *and* x_B^h *denote restrictions of the exact solution of problem (1) to* I_h *in case* A *and in case* B *, respectively, and let* x_h^A *and* x_h^B *denote solutions to discretisations in case* A *and* B*, respectively. Then*

$$(22) \qquad \|x_A^h - x_h^A\|_\infty \leq C \begin{cases} 1/n_o + h_A \ , \text{when} \ \ b,f \in C^1[0,1], \ ad \geq 1 \\ 1/n_o^2 + h_A \ , \text{when} \ \ b,f \in C^2[0,1], \ ad \geq 2 \end{cases},$$

$$(23) \qquad \|x_B^h - x_h^B\|_\infty \leq C(1/n_o^2 + h_B^2) \ , \quad \text{when} \ \ b,f \in C^2[0,1], \ ad \geq 2.$$

P r o o f: From (20) we have

$$\|x_A^h - x_h^A\|_\infty \leq \|(A_h + B_h)^{-1}\|_\infty \ \|r\|_\infty \ ,$$

and using the previous theorem and Theorems 1. and 2. we conclude (22).

Using Theorem 3. we prove (23) for case B.

6. REMARKS

1. We use a symmetric mesh and even n just to show the main ideas of the paper in a simpler manner.

2. Even the function $\lambda(s)$ need not be centrally symmetric. We can take $\lambda_1(s)$ for $s \in [0,0.5]$ with $q = q_1$, $a = a_1$ and $\lambda_2(s)$ for $s \in [0.5,1]$ with $q = q_2$, $a = a_2$, where $a_1 b(0)$, $a_2 b(1) \geq k$, $k=1,2$.

3. If we take $t_i = \lambda(i/n)$, we obtain Bahvalov's result |1|.

4. It is naturally more interesting to consider the case $s_\varepsilon \le \alpha$. When we take $\tau(\varepsilon) = \lambda(\alpha_2)$, we have $s_\varepsilon = \alpha_2 \le \alpha$.

Besides, for each ε there is $k=2,3,\ldots$, such that $s_\varepsilon \le \alpha_2 \le \alpha$ when we take $\tau(\varepsilon) = \varepsilon^{(k-1)/k}$.

5. The error of the numerical method gets smaller when n grows, but in such a way that n_o grows and the maximal step od difference schemes decreases outside the boundary layers.

7. NUMERICAL EXAMPLE

We shall illustrate the theoretical results with some computational results for the problem

$$-\varepsilon^2 x'' + x = 1, \quad x(0) = x(1) = 0$$

with solution

$$x(t) = 1 - \frac{\exp((t-0.5)/\varepsilon) + \exp((0.5-t)/\varepsilon)}{\exp(0.5/\varepsilon) + \exp(-0.5/\varepsilon)} .$$

This example represents a linear model of a catalytic reaction and it has been numerically treated in |2|, |6|, |8|.

Since $x(t)=x(1-t)$, we shall find the numerical solution only for $t \in [0,0.5]$. The number of mesh points in $(0,0.5]$ is denoted as before by m. The number m_s denotes how many mesh points are in $I_h \cap (0,\varepsilon]$.

In tables I, II and III we give the results for m=20, q=0.4, a=2. The values

$$g_s = \max \left\{ \left| \frac{x(t_i)-x_h(t_i)}{x_h(t_i)} \right| : t \in I_h \cap (0.\varepsilon] \right\}$$

and

$$g = \max \left\{ \left| \frac{x(t_i)-x_h(t_i)}{x_h(t_i)} \right| : t_i \in I_h \right\} ,$$

where $x(t)$ denotes the exact solution and $x_h(t)$ denotes the numerical solution to the given problem, are given in % .

Table I contains the numerical results for case A and for case B. Number k is the smallest integer which satisfies

$$\tau(\varepsilon) := \varepsilon^{(k-1)/k} \leq 1.5\,\varepsilon \ .$$

We used the meshes with points $t_i = \lambda(Ki)$, $i=1,2,\dots,n_o+1$, and t_i, $i=n_o+2,\dots,m-1$, which are given in Table II.

For $\varepsilon = 2^{-9}$, 2^{-13}, 2^{-17} in case B it is not possible to keep the same number (nine) of mesh points in $(0,\varepsilon]$, because of the conditions that the mesh satisfy (2. Case B). But we used the knots t_{n_o+2} and t_{n_o+3} with the property

$$\varepsilon \leq t_{n_o+2} - t_{n_o+1} \leq t_{n_o+3} - t_{n_o+2} \leq h_B \ ,$$

and a non-equidistant three-point scheme at mesh points t_{n_o+1}, t_{n_o+2}. Then estimate (23) is still valid, since for a three-point scheme we have

$$|r_i| \leq C\cdot\varepsilon \max\,(t_{i+1}-t_i, t_i-t_{i-1}) \leq h_B^2 \ , \ i=n_o+1, n_o+2 \ ,$$

when $ad \geq 2$.

Table III shows the numerical results for the case $t_i = \lambda(i/m)$ (as in |1|).

TABLE I

ε	k	n_o	m_s	Case A g_s%	Case A g%	Case B g_s%	Case B g%
2^{-5}	9	12	9	0.226	0.414	0.126	0.659
2^{-9}	16	12	9	0.596	1.388	2.402	3.974
2^{-13}	23	12	9	1.033	2.200	4.868	7.975
2^{-17}	30	12	9	1.385	2.232	7.548	12.175

TABLE II, Case A

	$\varepsilon=2^{-5}$	$\varepsilon=2^{-9}$	$\varepsilon=2^{-13}$	$\varepsilon=2^{-17}$
t_{14}	5.97 E-2	3.71 E-3	2.51 E-4	1.42 E-5
t_{15}	7.30 E-2	5.08 E-3	4.61 E-4	2.61 E-5
t_{16}	9.57 E-2	9.17 E-3	1.41 E-3	1.23 E-4
t_{17}	1.34 E-1	2.14 E-2	5.68 E-3	9.19 E-4
t_{18}	2.00 E-1	5.82 E-2	2.49 E-2	7.45 E-3
t_{19}	3.11 E-1	1.69 E-1	1.11 E-1	6.10 E-2

TABLE II, Case B

	$\varepsilon=2^{-5}$	$\varepsilon=2^{-9}$	$\varepsilon=2^{-13}$	$\varepsilon=2^{-17}$
t_{14}	5.78 E-2	7.16 E-3	6.92 E-4	7.38 E-5
t_{15}	6.85 E-2	2.26 E-2	1.63 E-2	1.57 E-2
t_{16}	8.77 E-2	3.80 E-2	3.19 E-2	3.13 E-2
t_{17}	1.22 E-1	6.88 E-2	6.31 E-2	6.26 E-2
t_{18}	1.85 E-1	1.30 E-1	1.26 E-1	1.25 E-1
t_{19}	2.97 E-1	2.54 E-1	2.50 E-1	2.50 E-1

TABLE III

ε	m_s	g_s%	g%
2^{-5}	6	0.130	0.119
2^{-9}	6	0.132	0.220
2^{-13}	6	0.134	0.297
2^{-17}	6	0.135	0.389

From the numerical results we can conclude that g_s and g do not change much when ε decreases, i.e. the convergence of given difference schemes is uniform in ε. The results from Table III are determined by ε, q, a and m. Table I shows that it is possible to achieve a greater number m_s, than in

Bahvalov's case. However, g_s and g are greater, but still tolerable, especially g_s - which is more interesting.

These are only some possibilities of mesh construction. The automatical construction of an optimal mesh has not been considered.

REFERENCES

|1| Bahvalov, A.S., *K oprimizacii metodov rešeniya kraevyh zadač pri naličii pograničnogo sloya, Ž. vyčisl.mat. i mat. fiz., T9. No 4, 841-859, 1969.*

|2| Bohl, E., *Finite Modelle gewöhnlicher Randwertaufgaben, B.G.Teubner, Stuttgart, 1981.*

|3| Bohl, E., J. Lorenz, *Inverse monotonicity and difference schemes of higher order.A summary for two- point boundary value problems, Aequ.Math. 19,1-36 , 1979.*

|4| Doolan,E.P.,J.J.H.Miller,W.H.A.Schilders, *Uniform numerical methods for problems with initial and boundary layers, Bode Press, Dublin, 1980.*

|5| Herceg,D., *Nichtäquidistante Diskretisierung der Grenzschichtdifferentialgleichungen und einige Eigenschaften von diskreten Analoga, Zbornik radova Prirodno- matematičkog fakulteta u Novom Sadu knjiga 9 (1979), 199-219.*

|6| Herceg,D., *O korišćenju neekvidistantne mreže kod diferencnih postupaka, Zbornik radova Prirodno-matematičkog fakulteta u Novom Sadu, knjiga 10 (1980), 102-112*

|7| Lorenz,J., *Zur Theorie und Numerik von Differenzenverfahren für singuläre Störungen, Habilitationsschift, Konstanz, 1980.*

|8| Reinhardt,H.-J., *A posteriori Error Estimates for the Finite Element Solution of a Singularly Perturbed Linear Ordinary Differential Equation, SIAM J.Num. Anal. 18, 406-430 (1981).*

|9| Varah, J.M., *A Lower Bound for the Smallest Singular Value of a Matrix, Linear Algebra Appl., 11, 3 - 5, 1975.*

REZIME

NEKE DIFERENCNE ŠEME ZA SINGULARNI PERTURBACIONI PROBLEM NA NEEKVIDISTANTNOJ MREŽI

U radu se posmatra diskretizacija problema (1) na neekvidistantnoj mreži (4). Pri tom se mreža I_h formira tako da diskretni analogoni za (1), dobijeni primenom operatora L_h^1, L_h^2, i L_h^3, slučaj A i slučaj B, imaju jedinstvena rešenja koja uniformno po ε konvergiraju ka rešenju problema (1) kada $n_o \to \infty$ i $h_A \to 0$, odnosno $h_B \to 0$. Kada se čvorovi t_i mreže I_h odredjuju prema $t_i = \lambda(i/n)$, $i=0,1,\dots,n$, dobija se poznati rezultat Bahvalova |1|. Slobodniji izbor čvorova mreže koji se predlaže u ovom radu omogućava da, u odnosu na mrežu Bahvalova, veći broj čvorova leži u uskom graničnom sloju, pri istom ukupnom broju tačaka mreže.