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ON ALTERNATIVE NONSTATIONARY ITERATIVE PROCEDURES

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In this paper the equation

(1) $x = Tx + f \ , \ f \in B$

is considered in a Banach's partially ordered space B. It is supposed that T is a linear operator and that

(2) $u \leq v \Rightarrow Tu \geq Tv \ , \ (u,v \in B)$.

To solve equation (1) a nonstationary iterative procedure is used

(3) $$\begin{cases} z_n = T_n z_{n-1} + f & (n=1,2,\dots, \text{ or } n=2,3,\dots) \\ T_n x = Tx + \rho_n \ , & \rho_n \in B, \quad (z_o, z_1 \in B) \ . \end{cases}$$

"Auxiliary" operators A_k and B_k are introduced so that

$$B_k x \leq \rho_k \leq A_k x$$

where x is a fixed element of space B which is expressed via sequence (3). In this way it is possible to construct invariant intervals for operator T'.

(4) $T'x = Tx + f$

which allows an a posteriori error estimation and an acceleration of the procedure (3). The achieved results represent a generalization of the results from |1| concering the alternative iterative sequences. In Lemma1 the idea of proving the statement 1.2 |1| was used. For the sequence z_k it cannot be stated that it converges to the solution of equation (1), but for each k, fixed by the conditions of the following theorems, it is possible to determine a neighbourhood of the point x^*, x^*- being the solution of equation (1), to which z_k belongs. Furthermore , it is possible to determine the point from that neighbourhood which represents a better approximation for x^* than z_k is. The neighbourhood diametar depends on the operators A_k and B_k and can be made small up to the extent to which we are able to determine these operators. Let us list some notations which will be used in this paper.

(5) $$x_n = Tx_{n-1} + f, \ x_o \in B \quad (n=1,2,\dots)$$

(6) $$\begin{cases} \delta z_{2i} = z_{2i} - z_{2i-1} & (i=1,2,\dots) \\ \delta z_{2i+1} = z_{2i} - z_{2i+1} & (i=0,1,\dots) \end{cases}$$

(7) $$Gz_k(s) = z_k + s\delta z_k$$

(8) $$Iz_k(s,t,p,q) = [Gz_k(s) + t_k \ , \ Gz_k(p) + q_k]$$

(9) $$\begin{cases} \Phi_k(A,B) = -T_{k-1}A_k z_{k-2} + B_k z_{k-1} + (B_k A_k + B_k) u_k - B_k A_k z_{k-2} \\ \Phi_k'(A,B) = -T_{k-1}B_k z_{k-2} + A_k z_{k-1} + (A_k^2 + B_k) u_k + A_k B_{k-2} \end{cases}$$

where

(10) $$\begin{cases} u_k = z_{k-1} - (A_k + B_k) z_{k-2} & \text{for } k = 2i \\ u_k = z_{k-2} & \text{for } k = 2i+1 \end{cases}$$

(11) $$\begin{cases} \phi_k(A,B) = (B_k - A_k) z_{k-1} + (B_k A_k + A_k B_k) u_k - B_k A_k z_{k-2} + A_k f \\ \phi_k'(A,B) = (A_k - B_k) z_{k-1} + (A_k^2 + B_k^2) u_k + A_k B_k z_{k-2} + B_k f \end{cases}$$

LEMMA 1. *Let the equation (1) be given in* B *with linear monotonously non-increasing operator* T *which satisfies condition (2). Let for some* j *in sequence (5) there exist* $p_i, q_i \in B$ *so that it holds that*

(12) $p_i \le \delta x_i \le q_i \quad (i=j-1,j)$ *and*

(13) $$\begin{cases} p_j \ge s_j q_{j-1} + S_j q_j \\ q_j \le S_j p_{j-1} + s_j p_j \end{cases}$$

where s_j *and* S_j *are real numbers and*

(14) $$0 < s_j \le S_j$$

Then, for $j=2i$ *operator* T' *maps the interval* $Ix_j(-S,0,-s,0)$ *into itself and for* $j=2i+1$ *the same operator maps the interval* $Ix_j(s,0,S,0)$ *into itself.*

P r o o f. Let $j=2i$. From (12) and (13) we get

(15) $$s_j\delta x_{j-1} + S_j\delta x_j \le \delta x_j \le S_j\delta x_{j-1} + s_j\delta x_j$$

By applying operator T onto the inequality (15), because of $\delta x_j = -T\delta x_{j-1}$, we get

$$x_j - s_j\delta x_j \ge T(x_j - S_j\delta x_j) + f$$
$$v_o \ge Tu_o + f = v_1$$
$$v_o = x_j - S_j\delta x_j$$
$$u_o = x_j - s_j\delta x_j$$

From (15) it also follows that

$$u_o \le Tv_o + f = u_1$$

Because of (14) $u_o \le v_o$ so that

$$u_o \le u_1 \le v_1 \le v_o$$

For an arbitrary $s \in [u_o, v_o]$ it holds that

$$u_1 = Tv_o + f \le Ts + f \le Tu_o + f = v_1$$

Hence, operator T' leaves the interval $Ix_j(-S,0,-s,0)$ invariant. In a similar way, the statement for $j=2i+1$ can be proved.

COROLLARY 1. *The quantities* p_i, q_i $(i=j,j-1)$ *determined in Lemma 1 are nonnegative.*

THEOREM 1. *Let the linear monotonously nonincreasing operator* T *be defined in* B. *Let for some* $k \geq 2$ *in sequence (3) the following hold*

1.1. $(T-A_k)u_k \leq T_n u_k \leq (T+B_k)u_k \qquad (n=k,k-1)$

$$u_k = z_{k-2},\ z_{k-1} - (A_k+B_k)z_{k-2}$$

1.2. There are $g_k, G_k \in R$ *such that*

$$0 < g_k \leq G_k$$

a). for $k=2i$

$$(1-G_k)\delta z_k - g_k\delta z_{k-1} \geq (A_k+g_kB_k+G_kB_k)w_k + \Phi_k(A,B) + G_k \cdot \Phi_k'(A,B)$$

$$G_k\delta z_{k-1} - (1-g_k)\delta z_k \geq (B_k+g_kA_k+G_kA_k)w_k + \Phi_k'(A,B) + g_k\Phi_k(A,B)$$

$$w_k = z_{k-1} - (A_k+B_k)z_{k-2}$$

b) $k=2i+1$

$$(1-G_k)\delta z_k - g_k\delta z_{k-1} \geq (B_k+g_kA_k+G_kA_k)z_{k-2} + \Phi_k'(A,B) + G_k\Phi_k(A,B)$$

$$G_k\delta z_{k-1} - (1-g_k)\delta z_k \geq (A_k+g_kB_k+G_kB_k)z_{k-2} + \Phi_k(A,B) + g_k\Phi_k'(A,B)$$

Then, in the case $k=2i$, *operator* T' *maps the interval* $Ix_2(-S,0,-s,0)$ *into itself and, when* $k=2i+1$, *operator* T' *maps the interval* $Ix_2(s,0,S,0)$ *into itself. In addition* $s_2=g_k, S_2=G_k$, *and* x_n *is determined by (5) for* $x_o=z_{k-2}$.

P r o o f. For $k=2i$ we shall determine the quantites p_i, q_i $(i=1,2)$ which satisfy the conditions of Lemma 1 for x_n determined in the above way. Let us introduce the notations

$$(16)\qquad \begin{cases} \bar{z}_n = T_{k+n-2}\bar{z}_{n-1} + f, \quad \bar{z}_o = z_{k-2} \quad (n=1,2,\dots) \\ \ell_n = (T+(-1)^n(A_k)^{2-n}(B_k)^{n-1})\ell_{n-1} + f, \ \ell_o = \bar{z}_o \\ y_n = (T+(-1)^{n-1}(A_k)^{n-1}(B_k)^{2-n})y_{n-1} + f, \ y_o = \bar{z}_o \\ x_n = Tx_{n-1} + f, \quad x_o = z_{k-2} \end{cases}$$

$$(A_k)^o = (B_k)^o = E, \quad E \text{ the identity operator.}$$

According to condition 1.1

(17) $$\ell_1 \le \bar{z}_1 \le y_1$$

Let us show that

(18) $$\ell_2 \ge \bar{z}_2 \ge y_2$$

Because of 1,1 it follows that

$$\ell_1 = T\bar{z}_o + f + \rho_{k-1} - \rho_{k-1} - A_k\bar{z}_o \ge w_k$$

$$\ell_2 \ge T\ell_1 + B_k w_k + f \ge T\ell_1 + \rho_{k-1} + f \ge \bar{z}_2$$

$$\bar{z}_2 \ge T\bar{z}_1 - A_k\bar{z}_1 + f \ge (T-A_k)y_1 + f = y_2$$

On the basis of the above inequalities we get the following relation between x_n and $\bar{z}_k$

(19) $$\begin{cases} x_n = \bar{z}_n + (-1)^{n-1} a_n + (-1)^n k_n & (n=1,2\ ) \\ x_n = \bar{z}_n + (-1)^n b_n + (-1)^{n-1} c_n & (n=1,2\ ) \end{cases}$$

where

$$\begin{cases} x_n = y_n + (-1)^n k_n \\ \bar{z}_n = \ell_n + (-1)^{n-1} c_n \\ y_n = \bar{z}_n + (-1)^{n-1} a_n \\ \ell_n = \bar{z}_n + (-1)^n b_n \end{cases}$$

In view of (17) and (18) it holds that $a_i \ge 0$, $b_i \ge 0$, $(i=1,2\ )$. Since

(20) $$\begin{cases} T\bar{z}_o \le \bar{z}_1 - f + A_k w_k \\ -TH\bar{z}_o \le -T_{k-1}H\bar{z}_o + B_k w_k & (H=A_k,\ B_k) \end{cases}$$

(21) $$\begin{cases} c_2 = B_kT\bar{z}_o - TA_k\bar{z}_o - B_kA_k\bar{z}_o + B_kf \\ k_2 = A_kT\bar{z}_o - TB_k\bar{z}_o + A_kB_k\bar{z}_o + A_kf \end{cases}$$

after introducing (20) into (21) we get

$$(22)\qquad \begin{cases} c_2 \le \Phi_k(A,B) \\ k_2 \le \Phi_k'(A,B) \end{cases}$$

From (19), (22), (6) it follows that the quatities

$$(23)\qquad \begin{cases} p_1 = \delta z_{k-1} - A_k w_k \\ p_2 = \delta z_k - A_k w_k - \Phi_k(A,B) \\ q_1 = \delta z_{k-1} + B_k w_k \\ q_2 = \delta z_k + B_k w_k + \Phi_k'(A,B) \end{cases}$$

satisfy inequality (12) for j=2, while x_j is determined by (16).

The quantities (23) also satisfy inequalities (13) for $S_2=G_k$, $s_2=g_k$, hence after applying Lemma 1 the statement is proved.

In a similar way, it can be shown that for $k=2i+1$ the quantities

$$(24)\qquad \begin{cases} p_1 = \delta z_{k-1} - B_k z_{k-2} \\ p_2 = \delta z_k - B_k z_{k-2} - \Phi_k'(A,B) \\ q_1 = \delta z_{k-1} + A_k z_{k-2} \\ q_2 = \delta z_k + A_k z_{k-2} + \Phi_k(A,B) \end{cases}$$

satisfy inequalities (12) and (13) for j=2 and $S_2=G_k$ $s_2=g_k$.

COROLLARY 2. *Let condition 1.1 of Theorem 1 hold. Let* $Hu_k \ge 0$ $(H=A_k, B_k)$, *where* u_k *is determined by (10). Then,*

$$\Phi_k(A,B) \ge 0 \qquad \text{and} \qquad \Phi_k'(A,B) \ge 0 \ .$$

THEOREM 2. *Let the operators* A_k *and* B_k *be commutative with operator* T. *Theorem 1 is valid if* Φ_k *is replaced by* ϕ_k *and* Φ_k' *by* ϕ_k' .

P r o o f. The theorem can be proved in the same way as Theorem 1 but the following majorizations have to be used

$$c_2 \leq \phi_k(A,B)$$
$$k_2 \leq \phi'_k(A,B)$$

THEOREM 3. *Let a linear monotonously nonincreasing operator* T *be defined in* B. *Let for a* $k \geq 2$ *in sequence (3) it holds that*

3.1. $(T+A_k)u_k \leq T_n u_k \leq (T-B_k)u_k \quad (n=k,k-1)$

$$u_k = (z_{k-2},\ z_{k-1} - (A_k+B_k)z_{k-2})$$

3.2. There exist such real numbers g_k *and* G_k *that*

$$0 < g_k \leq G_k \qquad \text{and}$$

a) for $k=2i$

$$(1-G_k)\delta z_k - g_k \delta z_{k-1} \geq -(A_k+G_kB_k+g_kB_k)w_k + \Phi_k(B,A) - g_k\Phi'_k(B,A)$$
$$G_k\delta z_{k-1} - (1-g_k)\delta z_k \geq -(B_k+g_kA_k+G_kA_k)w_k - \Phi_k(B,A) - g_k\Phi'_k(B,A)$$

b) for $k=2i+1$

$$(1-G_k)\delta z_k - g_k\delta z_{k-1} \geq -(A_k+G_kB_k+g_kB_k)z_{k-2} - \Phi'_k(B,A) - G_k\Phi_k(B,A)$$
$$(1-g_k)\delta z_k - G_k\delta z_{k-1} \leq (B_k+G_kA_k+g_kA_k)z_{k-2} + g_k\Phi'_k(B,A) + \Phi_k(B,A)$$

Then, operator T′ *leaves invariant the interval* $Ix_2(-S,0,-s,0)$ *for* $k=2i$, *and interval* $Ix_2(s,0,S,0)$ *for* $k=2i+1$, *respectively. Here* $S_2=G_k$, $s_2=g_k$ *and* x_n *is determined by (16).*

P r o o f. The proof is similar to that for Theorem 1 where sequences ℓ_n and y_n are defined in the following way

$$\ell_n = (T+(-1)^{n-1}(A_k)^{2-n}(B_k)^{n-1})\ell_{n-1}+f, \quad \ell_o=z_{k-2} \quad (n=1,2)$$
$$y_n = (T+(-1)^n(A_k)^{n-1}(B_k)^{2-n})y_{n-1}+f, \quad y_o=z_{k-2}\ .$$

Afterwards it is necessary to show that for $k=2i$ the quantities

$$p_1 = \delta z_{k-1} + A_kw_k$$
$$q_2 = \delta z_{k-1} - B_kw_k$$

$$p_2 = \delta z_k - \Phi_k(B,A) - B_k w_k$$

$$q_2 = \delta z_k + \Phi_k'(B,A) + A_k w_k$$

and, for k=2i+1 the quantities

$$p_1 = \delta z_{k-1} + B_k z_{k-2}$$

$$q_1 = \delta z_{k-1} - A_k z_{k-2}$$

$$p_2 = \delta z_k + B_k z_{k-2} + \Phi_k(B,A)$$

$$q_2 = \delta z_k - A_k z_{k-2} - \Phi_k'(B,A)$$

satisfy inequalities (12) and (13) for $j=2$, $s_2=g_k$, $S_2=G_k$.

THEOREM 4. *Let the operators A_k and B_k be commutative with operator* T. *Then, Theorem 3 holds if Φ_k is replaced by ϕ_k and Φ_k' by ϕ_k'.*

COROLLARY 3. *If condition 3.1 is satisfied by $A_k u_k \le 0$ and $B_k u_k \le 0$, u_k is determined by (10), then,*

$$\Phi_k(B,A) \le 0 \qquad \textit{and} \qquad \Phi_k'(B,A) \le 0.$$

In the above theorems the invariant intervals for operator T' were determined. The interval boundaries were expressed in terms of a function of x_n and z_n. Using the relation between x_n and z_n we shall determine a somewhat wider interval $Iz_k(U,m,V,n)$, so that

$$Ix_2\ (u,0,v,0) \subset Iz_k(U,m,V,n) = Iz_k$$

We shall show that for k=2i

$$\begin{cases} m_k = -c + U_k(k+q_k') \\ n_k = k - V_k(c+p_k') \end{cases}$$

and for k=2i+1

$$\begin{cases} m_k = -c + U_k\ (-k-p_k') \\ n_k = k + V_k\ (c+q_k'), \text{ where} \end{cases}$$

$$U_k = u_2\ , \quad V_k = v_2\ ,$$

(25) $$\begin{cases} p_k' = \delta z_{k-1} - p_1 \\ q_k' = q_1 - \delta z_{k-1} \end{cases}$$

$$\begin{cases} c_2 \leq c \\ k_2 \leq k \end{cases} .$$

The functions which we can introduce for c and k as well as for p_1 and q_1 differ from the theorem to theorem and should be determined for each theorem separately.

For k=2i we have $U_k < 0$, $V_k < 0$ (Lemma 1.) and

$$\bar{z}_1 - q_k' \leq x_1 \leq \bar{z}_1 + p_k'$$
$$\bar{z}_2 - c \leq x_2 \leq \bar{z}_2 + k ,$$

so that

$$x_2 + U_k\delta x_2 \geq z_k - c + U_k(\delta z_k + k + g_k') = Gz_k(U) + m_k$$
$$x_2 + V_k\delta x_2 \leq z_k + k + V_k(\delta z_k - c - p_k') = Gz_k(V) + n_k .$$

For k=2i+1 we have $U_k > 0$, $V_k > 0$ and

$$\bar{z}_1 - p_k' \leq x_1 \leq \bar{z}_1 + q_k'$$
$$\bar{z}_2 - c \leq x_2 \leq \bar{z}_2 + k ,$$

so that

$$x_2 + V_k\delta x_2 \leq Gz_k(V) + n_k$$
$$x_2 + U_k\delta x_2 \geq Gz_k(U) + m_k .$$

THEOREM 5. *Let us suppose that by means of Theorem 1 or Theorem 3 the interval* Iz_k *and* Iz_{k+1} *are determined, and that*

5.1. $g_k \leq g_{k+1}$, $G_k \geq G_{k+1}$

5.2. $p_i' \geq 0$, $q_i' \geq 0$, $(i=k,k+1)$, *where* p_i' *and* q_i' *are determined by (25). Then,*

(26) $$Iz_k \subseteq Iz_{k+1}$$

P r o o f. We shall demonstrate only a part of the statement concerning Theorem 1. Theorem 3 can be proved in an analogous way. According to (23) and (24) for k=2i

$$A_k w_k \geq 0, \quad B_k w_k \geq 0, \quad A_{k+1} z_{k-1} \geq 0, \quad B_{k+1} z_{k-1} \geq 0 ,$$

hence because of Corollary 2 $\Phi_k'(A,B) \geq 0$ and $\Phi_k(A,B) \geq 0$.

Consider the difference between the lower limits of the intervals Iz_{k+1} and Iz_k :

$$z_{k+1}+g_{k+1}\delta z_{k+1}+m_{k+1}-z_k+G_k\delta z_k-m_k \geq -\delta z_{k+1}(1-g_{k+1})+G_{k+1}\delta z_k +$$
$$+ m_{k+1}-m_k = -\delta z_{k+1}(1-g_{k+1})+G_{k+1}\delta z_k-\Phi_{k+1}(A,B)-g_k(\Phi_{k+1}'(A,B) +$$
$$+ B_{k+1}z_{k+1})+\Phi_k(A,B)+G_k(\Phi_k'(A,B)+B_k w_k) \geq G_{k+1}\delta z_k-\delta z_{k+1}(1-g_{k+1}) -$$
$$- \Phi_{k+1}(A,B)-g_{k+1}(\Phi_{k+1}'(A,B)+B_{k+1}z_{k+1})+\Phi_k(A,B)+G_k(\Phi_k'+B_k w_k) \geq 0,$$

since, according to 1.2 b) for k+1=2i+1

$$G_{k+1}\delta z_k-(1-g_{k+1})\delta z_{k+1}-g_{k+1}(B_{k+1}z_{k+1}+\Phi_k'(A,B)) -$$
$$- \Phi_k(A,B) \geq A_{k+1}z_{k+1}+G_{k+1}B_{k+1}z_{k+1} \geq 0 .$$

For the upper limits

$$z_k-g_k\delta z_k+n_k-z_{k+1}-G_{k+1}\delta z_{k+1}-n_{k+1} \geq (1-G_{k+1})\delta z_{k+1} -$$
$$- g_{k+1}\delta z_k-G_{k+1}\Phi_{k+1}(A,B)-G_{k+1}A_{k+1}z_{k-1}-\Phi_k'(A,B) +$$
$$+ \Phi_k'(A,B) + g_k(\Phi_k(A,B)+A_k w_k) \geq 0 .$$

In this way we have obtained relation (26) for k=2i. The procedure is analogous for k=2i+1.

Relation (26) for the intervals determined by Theorems 2 and 4 can be obtained in a similar way, if it can be stated that k and c are nonnegative.

REMARK 1. When $A_k = B_k = 0$ and $B = R^n$, Theorem 1,2,3 and 4 are reduced to Theorem 1.2 |1|.

Experimental results. In |4| and |5| a nonstationary iterative procedure for solving the Fredholm integral equation of the second kind was described.

(27) $$u(s) = \int_a^b K(s,t)u(t)dt + f(s), \quad (s,t \in [a,b] = I)$$

$$f(s) \in C(I), \quad K(s,t) \in C(I \times I), \quad (a,b \in R).$$

The approximate solution $z_k(s)$ is determined from the formulas

(28) $$\begin{cases} z_o = f \\ z_1 = K_{m_1} z_o + f \\ z_k = K_{m_k} p_n r_n z_{k-1} + f, \quad (k=2,3,\dots) \end{cases}$$

where

a) r_n - operator, $r_n : C = C(I) \to R^n$, $r_n u = \{u(s_i)\}_{i=1}^n$

b) p_n - operator, $p_n : R^n \to C$, $p_n z = S_\Delta(z,s_j)$,

$S_\Delta(z,s_j)$ - third degree spline on the grid

$\Delta : a = s_1 < s_2 < \dots < s_n = b$

c) $K : C \to C$

$$Ku = \int_a^b K(s,t)u(t)dt$$

The operators K_m approximate the integral operator K and they arise as a result of replacement of the integral by some quadratic formula.

d) $K_m : C \to C$

$K_m u = \sum_{j=o}^{p} d_j(m)K(s,t_j)u(t_j)$, $p = 2^m + 1$ where $d_j(m)$ - weighting coefficient of the applied quadratic formula and m is determined by

$$\|r_n(K_m u - K_{m-1}u)\| \le \varepsilon, \quad \varepsilon > 0 \text{ given.}$$

The iterative procedure (28) has the from (3) if it is taken that

$$\rho_k = K_{m_k} p_n r_n z_{k-1} - K z_{k-1} \ .$$

In |4| and |5| the conditions were given which enable us to get the estimation of the form

(29) $$|\rho_k| \le \varepsilon_1 \ , \ \varepsilon_1 = \varepsilon_1(\varepsilon, \|\Delta\| \ , \ m_k, \ q) \ ,$$

where q is the order accuracy with which the calculations were caried out. If an estimation of the form (29) is possible, than Theorems 2 and 4 can be applied onto (28) where it has been taken that

(30) $$A_k = B_k = (\varepsilon_1 \ / \ (\min_t z_{k-2}(t))^{-1})E \ .$$

An application of Theorem 2 is demonstrated on Love´s equation

$$K(s,t) = \frac{-1}{\pi(1+(s-t)^2)} \ , \quad -1 \le s,t \le 1$$

$$f(s) = 1$$

The computation is stopped when for some p

$$|r_n \ (Z_p - z_{p-1})| \le \delta \ .$$

The extremes were determined by means of the cubic spline so that an error of the order $0(\|\Delta\|^3)$ has been introduced (|4|).

The results are shown in Table 1. DG_k denotes the lower interval limit obtained after the k-th integration and GG_k- the upper limit, while AS_k denotes the arithmetic mean of the obtained interval. For a comparison, we give the results of Brakhage (Über die numeriche Behandlung von Integralgleichungen nach der Quadraturformel Methode, Numer.Math. 2, 183-196, (1960)).

s_i	$\tilde{u}(s_i)$
0	0.6574172
0.25	0.6638282
0.5	0.6831709
0.75	0.7148688
1	0.7577358

where $|u^*(s_i) - \tilde{u}(s_i)| \le 0.0024$, $u^*(s)$ is the solution of equation (27).

TABLE 1.

		$\varepsilon_1 = 10^{-4}$		$\delta = 10^{-6}$		$\varepsilon = 10^{-5}$			
1	$DG_2(s_i)$	$GG_2(s_i)$	$AS_2(s_i)$	$DG_3(s_i)$	$GG_3(s_i)$	$AS_3(s_i)$	$z_3(s_i)$	$z_{18}(s_i)$	$z_2(s_i)$
1	.6566548	.6587830	.6577189	.6569388	.6578150	.6573770	.6249890	.6574104	.7288849
2	.6569104	.6569374	.6579740	.6571951	.6580708	.6576331	.6252747	.6576662	.7290754
3	.6576786	.6598005	.6587396	.6579645	.6588385	.6584015	.6261320	.6584349	.7296479
4	.6589601	.6610739	.6600170	.6592481	.6601195	.6596839	.6275620	.6597166	.7306042
5	.6607566	.6628587	.6618078	.6610472	.6619153	.6614814	.6295655	.6615236	.7319465
6	.6630678	.6651559	.6641119	.6633620	.6642258	.6637940	.6321411	.6638279	.7336788
7	.6659029	.6679728	.6669378	.6662002	.6670587	.6666296	.6352980	.6666608	.7358048
8	.6692557	.6713049	.6702805	.6695573	.6704090	.6699831	.6390278	.6700139	.7383280
9	.6731286	.6751542	.6741414	.6734343	.6742787	.6738565	.6433313	.6738865	.7412517
10	.6775193	.6795185	.6785190	.6778290	.6786654	.6782472	.6482043	.6782765	.7445774
11	.6824231	.6843927	.6834080	.6827388	.6835661	.6831524	.6536431	.6831827	.7483046
12	.6878309	.6897686	.6887999	.6881499	.6889672	.6885586	.6596296	.6885879	.7524297
13	.6937294	.6956322	.6946809	.6940510	.6948576	.6944544	.6661501	.6944826	.7569458
14	.7001002	.7019658	.7010331	.7004232	.7012186	.7008209	.6732827	.7008481	.7618415
15	.7069178	.7087445	.7078311	.7072418	.7080252	.7076335	.6806993	.7076597	.7671001
16	.7141521	.7159381	.7150452	.7144756	.7152464	.7148612	.6886642	.7148862	.7726998
17	.7217648	.7235084	.7226367	.7220736	.7228317	.7224526	.6970160	.7224901	.7786131
18	.7297127	.7314122	.7305642	.7300320	.7307768	.7304044	.7057638	.7304242	.7848067
19	.7379453	.7395997	.7387726	.7382593	.7389903	.7386248	.7147927	.7386456	.7912426
20	.7464106	.7480192	.7472150	.7467139	.7474360	.7470777	.7240670	.7470963	.7978809
21	.7550457	.7566085	.7558272	.7553496	.7560527	.7557013	.7335219	.7557197	.8046689
	g_k		0.3069177			0.3089443			
	G_k		0.3149267			0.3145148			
	$\max(GG_k(s_i)-DG_k(s_i))/2$		0.0010140			0.0004381			

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REZIME

O ALTERNATIVNIM NESTACIONARNIM ITERATIVNIM POSTUPCIMA

U |1| su date teoreme koje omogućavaju ubrzanje stacionarnih iterativnih postupaka i daju aposteriornu ocenu greške. U ovom radu su dokazane analogne teoreme za nestacionarne iterativne postupke.