Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 11 (1981)
Review of Research Faculty of Science-University of Novi Sad, Volume 11 (1981)

ON AN A POSTERIORI ERROR ESTIMATION IN SOLVING SOME CLASSES OF OPERATORS EQUATIONS

Katarina Surla
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

In |7| and |8| some possibilities were described for determining invariant intervals for positive linear operators by means of nonstationary iterative methods. Here the application of these results will be demonstrated for approximate solutions of the systems of integral and linear equations. In this way important information about the equation solution can be obtained on the basis of two iterations alone. At the same time, an acceleration of the iterative procedure is achieved.

We shall first define some operators and list notations and theorems which will be used in the present paper. The integral operator $K : C(I) \to C(I)$, $I=[a,b]$

(1) $$Ku = \int_a^b K(s,t)u(t)dt , \quad u \in C(I).$$

The restriction operator $r_n : C(I) \to R^n$

$$r_n u = \{u(s_i)\}_{i=1}^n , \quad u \in C(I) .$$

The prolongation operator $P_n : R^n \to C(I)$

$$p_n z = S_\Delta(z,s) , \quad z \in R^n ,$$

where $S_\Delta(z,s)$ is a third order spline on the grid

$$\Delta : a = s_1 < s_2 < \dots < s_n = b$$

with the ordinates z_i ($i=1,2,\dots,n$).

The operators K_m and K'_m approximate the operator K and they are obtained if the integral in (1) is substituted by the Newton-Cotes quadrature formula.

$$K_m,\ K'_m : C(I) \to C(I)$$

$$K_m u = \sum_{j=o}^{2^m+1} d_j(m) K(s,t_j) u(t_j),$$

where m is determined by

$$\| r_n (K_m u - K_{m-1} u) \| \le \sigma \ , \ \sigma > 0 \ , \text{ is given,}$$

$$K'_m u = \sum_{j=o}^{2^m+1} d_j(m) K(s,t_j) u(t_j),$$

where m is determined by

$$\| K'_m u - K'_{m-1} u \| \le \sigma \ , \quad \sigma > 0 \quad \text{is given.}$$

$$T'x = Tx + f$$

$$\delta z_k = z_k - z_{k-1}$$

$$Gz_k(s) = z_k + s\delta z_k$$

$$Iz_k(s,t,p,q) = [Gz_k(s) + t_k, \quad Gz_k(p) + q_k]$$

$$R_k(A,B) = T_{k-1} B_k z_{k-2} + B_k z_{k-1} + (B_k A_k + B_k^2 + A_k) z_{k-2}$$

$$R'_k(A) = T_{k-1} A_k z_{k-2} + A_k (z_{k-1} + z_{k-2})$$

$$r_k(A,B) = 2B_k z_{k-1} + 2B_k A_k z_{k-2} + B_k^2 z_{k-2} - B_k f$$

$$r'_k(A) = 2A_k z_{k-1} + A_k^2 z_{k-2} - A_k f$$

$$\| u \| = \max_{t \in I} |u(t)| \ , \quad u \in C(I)$$

$$\| z \| = \max_{1 \le i \le n} \| z_i \| \ , \quad z \in R^n$$

$$w_2(f,h) = \sup_{|t| \le h} \ \sup_{x-t,t,x+t \in [a,b]} |f(x+t) - 2f(x) + f(x-t)|$$

B - Banach's partially ordered space,

E - identity operator.

THEOREM 1. [8] *Let the linear positive operator* T *be defined in* B *and let for some* $k \ge 2$ *in the sequence*

$$(2)\qquad \begin{cases} z_n = T_n z_{n-1} + f\ , & (f, z_o \in B) \\ T_n x = Tx + \rho_n\ , & (\rho_n \in B) \end{cases}$$

the following hold:

There exist positive linear operators A_k and B_k such that

1.1. $(T-A_k) z_{k-2} \leq T_n z_{k-2} \leq (T+B_k) z_{k-2}$ $(n=k-1, k)$

1.2. $\delta z_{k-1} > (A_k+B_k) z_{k-2}$;

1.3 There exist real numbers s_k, S_k such that

a) $0 < s_k < S_k$

b) $\delta z_k - s_k(\delta z_{k-1} - \delta z_k) \geq (1+2s_k) A_k z_{k-2} + (1+s_k) r_k(A,B)$

c) $S_k(\delta z_{k-1} - \delta z_k) - \delta z_k \geq (1+2S_k) B_k z_{k-2} + (1+S_k) r_k'(A)$.

Then the operator T' maps the interval $Ix_2(\mu, 0, \eta, 0)$ into itself. Simultaneously $\mu_2 = s_k$, $\eta_2 = S_k$ and

$$x_n = Tx_{n-1} + f,\quad x_o = z_{k-2} \qquad (n=1,2,\ldots,\ )$$

In |8| it was supposed that the value of operator T and, consequently, the sequence x_n can not be calculated exactly. For this reason by the sequence (2) the wider interval (3) is to be determined.

$$(3)\qquad Iz_k(U,a,V,b) \supseteq Ix_2(\mu,0,\eta,0)\ , \text{ where}$$

$$\mu_2 = U_k\ ,\quad \eta_2 = V_k\ ,$$

$$a_k = -(1+U_k) R_k(A,B) - U_k A_k z_{k-2}$$

$$b_k = (1+V_k) R_k'(A) + V_k B_k z_{k-2}$$

THEOREM 2. |8| *If the operators* A_k *and* B_k *are commutative with* T, *then in Theorem 1 it is possible to replace* $R_k(A,B)$ *with* $r_k(A,B)$ *and* $R_k'(A)$ *with* $r_k'(A)$.

THEOREM 3. |8| *Let the intervals* Iz_k *and* Iz_{k+1} *be determined by Theorem 1 and let*

3.1. $s_k \le s_{k+1}$, $S_k \ge S_{k+1}$

3.2. $A_k z_{k-2} \ge 0$, $B_k z_{k-2} \ge 0$.

Then $Iz_{k+1} \subseteq Iz_k$

1. We shall now consider the integral equation

(4) $u = Ku + f, \quad a,b \in R$.

In order to find an approximate solution of equation (4), we shall use the nonstationary iterative procedure

(5) $z_k = K'_{m_k} z_{k-1} + f , \quad z_o = f \quad (k=1,2,\dots)$.

THEOREM 4. *Let for solving equation (4) a nonstationary iterative procedure (5) be applied with the Newton-Cotes quadrature formula, which is exact for polynoms of the order* $p, p \le \ell$.

Let

4.1 $K(s,t) \ge 0, \quad K(s,t) \in C^{\ell+1}(I \times I)$

4.2 $f(s) > 0, \quad f(s) \in C^{\ell+1}(I)$

4.3 $\frac{\partial^{\ell+1}}{\partial^{\ell+1}} [K(s,t) z_p(t)]$ $(p=k-1,k-2)$, *which does not change the sign on* I.

Let for some $k \ge 2$ *in sequence (5) it hold:*

4.4 $\delta z_{k-1} > 2\varepsilon z_{k-2}$, *where*

(6) $\varepsilon \ge \sigma (\min_t z_{k-2}(t))^{-1}$

4.5 $0 < m \le M < 1$, *where*

$m = \min_t m(t,\varepsilon), \quad M = \max_t M(t,\varepsilon)$,

$m(t,\varepsilon) = (\delta z_k(t) - F(t))(\delta z_{k-1}(t) + \varepsilon z_{k-2}(t))^{-1}$

$$M(t,\varepsilon) = (\delta z_k(t) + F(t) - 2\varepsilon^2 z_{k-2}(t))\cdot(\delta z_{k-1}(t) - \varepsilon z_{k-2}(t))^{-1}$$
$$F(t) = 2\varepsilon z_{k-1}(t) + (3\varepsilon^2+\varepsilon)z_{k-2}(t) - \varepsilon f(t)$$

Then, equation (4) has a solution $u^*(s)$ *in the interval* $Iz_k(s, t, S, q)$, *where*

$$s_k = m_k/(1-m_k), \qquad S_k = M_k/(1-M_k)$$
$$t_k = -(1+s_k)(2\varepsilon z_{k-1}(t)+\varepsilon^2 z_{k-2}(t)-\varepsilon f(t))+s_k\varepsilon z_{k-2}(t)$$
$$q_k = (1+S_k)(2\varepsilon z_{k-1}(t)+3\varepsilon^2 z_{k-2}(t)-\varepsilon f(t))+S_k\varepsilon z_{k-2}(t) .$$

P r o o f. Since $f>0$ then $z_{k-2}>0$ $(k\geq 2)$. According to Theorem 3.1 |6|

$$\|Kz_i - K_{m'_{i+1}} z_i\| \leq \sigma .$$

If we put $A_k = B_k = \varepsilon E$, ε being determined by (6), then onto the iterative sequence (5) Theorem 2 can be applied from which follows the existence of an invariant interval. The existence of a solution can be obtained from the monotony of the stationary sequence formed as in Theorem 2 or from Shauder´s fixed point theorem.

For an application of Theorem 2 on the nonstationary procedure of the form

$$r_n z_1 = r_n K_{m_1} f + r_n f$$
$$r_n z_{k+1} = r_n K_{m_k} p_n r_n z_k + r_n f , \quad (k=1,2,\dots)$$

see |9| and |7|.

2. In |5|, the solving the system of integral equations was described

$$\text{(7)} \qquad Y = \lambda \tilde{K} Y + F$$

where Y and F are vector functions

$$Y = \{y^i(t)\}_{i=1}^{n} \quad , \quad F = \{f^i(t)\}_{i=1}^{n}$$

and $\tilde{K}$ a matrix $n\times n$ whose element are operators K_{ij},

$$K_{ij} : C(I) \to C(I)$$

$$K_{ij}x = \int_a^b K_{ij}(s,t)x(t)dt$$

It was assumed that $f^i(t)$ and $K_{ij}(s,t)$ are continuous and periodic functions with a period b-a for all variables. For solving system (7) an iterative procedure is to be applied

$$(8) \qquad Y_k = F + \lambda \tilde{K} S_\Delta (Y_{k-1},t) \;, \quad Y_o = F \quad (k=1,2,\dots)$$

where

$$S_\Delta(Y_{k-1},t) = \{S_\nabla(y^i_{k-1},t\}_{i=1}^{n} .$$

For this a periodic cubic spline and an equidistant grid were used. It has been shown (|5|) that procedure (8) converges if

$$\frac{1}{4}(1+3\sqrt{3})|\lambda|K_o n(b-a) < 1$$

$$K_o \geq \max_{s,t} |K_{ij}(s,t)| \;, \quad (i\;,j=1,2,\dots,n) \;.$$

If the integral operators in (7) are positive, then Theorem 1 can be applied to procedure (8). Note that operator $K_{ij}S$ defined as

$$K_{ij}Sx = \int_a^b K_{ij}(s,t)S_\Delta(x,t)dt \;,$$

is not linear. Let us define the operators T_k and T_{k-1} in this way:

$$(9) \qquad \begin{aligned} T_j Y &= \lambda \tilde{K} Y + \rho_j \;, \quad \rho_j = \lambda \tilde{K} R_{j-1} \quad (j=k,k-1) \\ R_j &= S_\Delta(Y_j,t) - Y_j \;. \end{aligned}$$

According to Theorem 1 in |5|, it holds that :

$$-(1+\frac{2}{3\sqrt{3}})W_2(Y_j,h) \leq R_j \leq (1+\frac{2}{3\sqrt{3}})\; W_2(Y_j,h) \;,$$

$$W_2(Y_j,h) = \{w_2(y^i_j,h)\}_{i=1}^{n} .$$

Furthermore

$$(10) \qquad \begin{aligned} &-C\bar{K}W_2(Y_j,h) \leq \lambda \tilde{K} R_j \leq C\bar{K}W_2(Y_j,h), \text{ where} \\ &\bar{K} = \{\bar{K}_{ij}\}_{i,j=1}^{n} \\ &\bar{K}_{ij} = \max_{s,t} \int_a^b K_{ij}(s,t)dt, \quad C = \lambda(1+\frac{2}{3\sqrt{3}}) . \end{aligned}$$

Before proceeding to further Theorems we shall introduce some mew notations

$H = C\psi$, a ψ matrix $n \times n$ with the elements ψ_{ij},

$$\psi_{ij} = \bar{K}_{ij} \max \left[\bar{w}_2(y^i_{k-1},h)\ w_2(y^i_{k-1},h)\right] \left(\min_t |y^i_{k-2}(t)|\right)^{-1}$$

$$\Phi_k(t) = T_{k-1}Hy^i_{k-2}(t) + H^i y_{k-1}(t) + H(2H+E)y^i_{k-2}(t)$$

$$m^i_k(t,H) = (\delta y^i_k(t) - Hy^i_{k-2}(t) - \Phi_k(t)(\delta y^i_{k-1}(t) + Hy^i_{k-2}(t))^{-1}$$

$$M^i_k(t,H) = (\delta y^i_k(t) + Hy^i_{k-2}(t) + \Phi_k(t) - 2H^2 y^i_{k-2}(t)) \cdot (\delta y^i_{k-1}(t) - Hy^i_{k-2}(t))^{-1}$$

THEOREM 5. *Let for solving system (7) the iterative procedure (8) be applied and let the above assumptions on the continuity and periodicity of the functions* $f^i(s)$ *and* $K_{ij}(s,t)$ *hold.*

Let $\cdot K_{ij}(s,t) \geq 0$, $f^i(s) > 0$ $(i,j=1,2,\dots,n)$

Let for somr $k \geq 2$ *in sequence (8) the following hold:*

a) $\delta y_{k-1} > Hy_{k-2}$, b) $0 < m \leq M < 1$,

$$m_k \leq \min_i \left(\min_t m^i_k(t,H)\right), \qquad M_k \geq \max_i\left(\max_t M^i_k(t,H)\right)$$

Then, system (7) has a solution within the interval

(11) $IY_k(U,a,V,b)$, *where*

$$U_k = \frac{m_k}{1-m_k},\ V_k = \frac{M_k}{1-M_k}$$ *and* a_k *and* b_k *are defined by (3) for*

(12) $A_k = B_k = H$.

P r o o f. By a direct application of Theorem 1 the existence of an invariant interval for the operator K'

$$K'Y = \tilde{K}Y + F$$

is obtained. For this the relations (9),(10) and (12) are to be used. The solution existence is obtained by means of Schauder's fixed point theorem. Interval (11) contains invariant interval determined by Theorem 1.

In a similar way, Theorem 3 can be applied to give the relation between intervals (11) determined by steps k and k+1.

3. Let the systems of linear equations be given by

$$(13)\qquad \begin{aligned} x &= Tx + f, \quad f \in R^n \\ z &= T^{*}z + f \end{aligned}$$

T and T* are matrices n x n with elements t_{ij} and t^{*}_{ij} respectively. Let us suppose that

$$t^{*}_{ij} = t_{ij} + r_{ij}$$

$$|r_{ij}| \leq \varepsilon$$

The following theorem gives a two-sided iterative procedure for tha approiximate solving of systems (13) via matrix T*.

THEOREM 6. *Let* $T \geq 0, T^{*} \geq 0$.

Let for some k *in the sequence*

$z_n = T^{*}z_{n-1} + f$, (n=1,...,) *the following inequalities hold*

a) $\delta z_{k-1} \geq 0$

b) $m_k \delta z_{k-1} \leq \delta z_k \leq M_k \delta z_{k-1}$

m_k *and* M_k *are real numbers such that*

$$0 < m_k \leq M_k < 1.$$

Let for some j *in the sequences*

$$(14)\qquad \begin{cases} v_j = T^{*}v_{j-1} + f, & v_o = t_k + M_k\delta z_k/(1-M_k) \qquad (j=1,2,\dots) \\ u_j = T^{*}u_{j-1} + f, & v_o = z_k + m_k\delta z_k/(1-m_k) \end{cases}$$

it hold that

$$(15)\qquad \begin{cases} v_{j-1} - v_j \geq \varepsilon D v_{j-1} \\ -u_{j-1} + u_j \geq \varepsilon D' u_{j-1}, \end{cases}$$

D *matrix* n x n , $d_{ij} = 1$ (i,j=1,2,...,n)

D′ *matrix* n x n ,

$$d'_{ij} = \begin{cases} d_{ij} & t^*_{ij} \geq \varepsilon \\ 0 & t^*_{ij} < \varepsilon \end{cases}$$

Then, system (13) has x^* *as a solution and the following estimation is valid*

$$u_j - \varepsilon D' u_{j-1} \leq x^* \leq v_j + \varepsilon D v_{j-1} .$$

P r o o f. If we introduce the notations

$$V_1 = (T^* + \varepsilon D)V_o + f , \quad V_o = v_{j-1}$$
$$U_1 = (T^* - \varepsilon D')V_o + f, \quad U_o = u_{j-1}$$

then, because of (15),

$$V_o - V_1 \geq 0, \quad U_1 - U_o \geq 0, \quad 0 \leq U_o \leq V_o .$$

Since

$$T^* - \varepsilon D' \leq T \leq T^* + \varepsilon D ,$$

after applying the Theorem (|3| p. 346), it follows that

$$U_1 \leq x^* \leq V_1$$

Because of (15), the statement is proved.

REMARK 1.

The conditions a) and b) determine the invariant interval for T (|2|). If such an interval could be determined in another way, then for u_o and v_o in (14) the limits of such an interval can be chosen.

REMARK 2.

The two-sided procedure (14) is performed with one matrix only, i.e. with the one for which is supposed to be given. In |3| a similar procedure is obtained for two different matrices.

The monotony is preserved while the conditions (15) resulting from the given matrix T are valid.

REMARK 3.

In Theorem 5 the errors are not taken into account either with which f was defined or the rounding errors. A control of these errors is possible if the following matrix instead of the matrix εD, is taken

$$\bar{D} = \varepsilon_j E + \varepsilon D \quad , \quad \text{where}$$

$$\varepsilon_j \geq \sigma_j \ (\min_i v^i_{j-1})^{-1}, \qquad |\rho^i_j| \leq \sigma_j$$

$\rho_j = \rho_j' + r$, ρ_j' are rounding errors, and r is the error by which the vector f was given. Instead of the matrix $\varepsilon D'$, the matrix G should be taken

$$g_{ii} = \begin{cases} \varepsilon_i' + \varepsilon & \text{for} \quad t^*_{ii} \geq \varepsilon + \varepsilon_i \\ 0 & \text{otherwise} \end{cases}$$

$$g_{ij} = \begin{cases} \varepsilon & \text{for} \quad t^*_{ij} \geq \varepsilon, \quad i \neq j \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_j' = \sigma_j \ (\min_i u^i_{j-1})^{-1}$$

REMARK 4.

The application of Theorem 1, 2 and 3 for solving a system of linear equations in the case when both the matrix and the vector are given by an error and when the rounding errors are present was demostrated in |7|. The ideas about the determination of operators A_k and B_k for such cases were given in |9| .

REFERENCES

|1| Alberg D Ž., Nilson E., Uolš D Ž., *Teorija splajnov i ee prilaženija, Moskva (1972).*

|2| Albrecht, J., *Fehlerschranken und Konvergenzbeschleunigung bei einer monotonen oder alternierenden Iterationfloge, Numer.Math. 4, 196-208, (1962).*

|3| Albrecht J., *Iterationsfolge und ihre Verwendung zur Lösung linearer Gleichungssysteme. Numer. Math. 3, 345-358 (1961).*

|4| Kurpel'N.S., *Dvustoronnie metody rešenija sistem uravnenij.Kiev (1975).*

|5| Poljakov, V.R., *Splajn-funkcii pri rešenii sistem linejnyh integral'nyh uravnenij, Metody količ. i kačas. isled. diff. i integr. uravn., Kiev (1975).*

|6| Rowland J.H., Miel G.J., *Exit criteria for Newton-Cotes quadrature rules. Siam J. Numer.Anal. Vol. 14, N 6, (decem. 1977).*

|7| Surla K., *Nestacionarne iterativne metode pri rešavanju operatorskih jednačina, Doktorska disertacija, Novi Sad (1980).*

|8| Surla K., *Determination of the invariant interval for the positive linear operators by a nonstationary iterative procedure, Publ.Inst.Math. T 30 (44), 1981, 177-187.*

|9| Surla K., *Aposteriorna ocena greške i ubrzanje nestacionarnih iterativnih postupaka kada je poznata jedna sopstvena vrednost operatora i odgovarajući sopstveni elemenat. Zbornik radova PMF, Univ. u Novom Sadu. 10, 123-136 (1981).*

REZIME

O APOSTERIORNOJ OCENI GREŠKE PRI REŠAVANJU NEKIH KLASA OPERATORSKIH JEDNAČINA

U |7| i |8| su date teoreme koje omogućavaju odredjivanje invarijantnog intervala za pozitivne linearne operatore primenom nestacionarnih iterativnih postupaka. Dokazi se zasnivaju na egzistenciji specijalnih pozitivnih operatora. U ovom radu je pokazana konstrukcija navedenih operatora i primena dobijenih rezultata na približno rešavanje sistema linearnih i integralnih jednačina.