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WEYL - OTSUKI SPACES OF THE SECOND AND THIRD KIND

Mileva Prvanović
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

1. INTRODUCTION

The basic objects of spaces defined and investigate by T.Otsuki |1| are as follows: a tensor field P of the type (1,1) ($\det(P^i_j)\neq 0$) and the coefficients $'\Gamma^i_{jk}$ and $''\Gamma_j{}^i{}_k$ of the connections $'\Gamma$ and $''\Gamma$ respectivey . These connections are the contravariant respective covariant part of the regular general connection Γ , i.e. $'\Gamma$ is the ordinary affine connection with the help of which is defined the covariant derivative of a contravariant vector:

$$D_k v^i = \left(\frac{\partial v^a}{\partial x^k} + {'\Gamma}^a_{sk} v^s\right) P^i_a \ ;$$

$''\Gamma$ is the ordinary affine connection with help of which is defined the covariant derivative of a covariant vector:

$$D_k v_j = \left(\frac{\partial v_a}{\partial x^k} - {''\Gamma}^s_{ak} v_s\right) P^a_j \ ,$$

while for the tensor v^i_{jt} for example, we have:

$$D_k v^i_{jt} = \left(\frac{\partial v^a_{bc}}{\partial x^k} + {'\Gamma}^a_{sk} v^s_{bc} - {''\Gamma}^s_{bk} v^a_{sc} - {''\Gamma}^s_{ck} v^a_{bs}\right) P^i_a P^b_j P^c_t \ .$$

The connections $'\Gamma$ and $''\Gamma$ are not independent; they satisfy the condition

$$\frac{\partial P^i_j}{\partial x^k} + {''\Gamma}^i_{ak} P^a_j - {'\Gamma}^a_{jk} P^i_a = 0. \tag{1.1}$$

This cindition is equivalent with

$$D_k Q^i_j = 0 \ ,$$

where $Q = P^{-1}$, i.e.

(1.2) $$P^i_j \, Q^s_j = P^s_j \, Q^i_s = \delta^i_j \ .$$

The Weyl-Otsuki space (W-O_n -space) is defined and investigated by A.Moór (|2|, |3|). This is an Otsuki space endowed with a symmetric positive definite metric tensor g_{ij} ($\det(g_{ij}) \neq 0$), and a recurrence vector γ_k such that the following conditions are satisfied:

a) the metric tensor is recurrent, i.e.

$$D_k g_{ij}(x) = \gamma_k(x) g_{ij}(x) \ ;$$

b) the covariant part $''\Gamma$ of the regular general connection Γ is symmetric; and

c) (1.3) $P_{ij} = g_{is}P^s_j = g_{js}P^s_i = P_{ji}$.

In W-O_n spaces, coefficients of connection $''\Gamma$ have the form |2|:

(1.4) $$''\Gamma^i_{jk} = \left\{ {i \atop j\ k} \right\} - \frac{1}{2}\, g^{is}(\gamma_j g_{ab} Q^a_k Q^b_s + \gamma_k g_{ab} Q^a_s Q^b_j - \gamma_s g_{ab} Q^a_j Q^b_k) ,$$

where $\left\{ {i \atop j\ k} \right\}$ are Christoffel symbols of the second kind with respest to the tensor g_{ij}. Substituting (1.4) into (1.1) we obrain the corresponding connection $'\Gamma$.

In this paper we investigate some differently defined Weyl-Otsuki spaces. In fact, we investigate the Otsuki space where condition c) is satisfied, and instead of conditions a) and b) - the following conditions are satisfied

a′) $D_k g_{ij}(x) = \gamma_k(x) m_{ij}(x)$,

where $m_{ij}(x)$ is a field of symmetric tensor;

b′) the contravariant part $'\Gamma$ of regular general connection is symmetric.

We have considered in |4| a special case of such $W\text{-}O_n$ spaces, namely the case $\gamma_k = 0$. Sime results obtained in |4| can be generalized for the general case a´). In fact, in exactly the same manner as in |4|, we find that the regular general connection satisfying a´), b´) and c) has the form

$$(1.5)\qquad {}''\Gamma^i_{jk} = {}''\overset{m}{\Gamma}{}^i_{jk} + \frac{1}{2} g^{si}(\gamma_q m_{pk} Q^q_s Q^p_j - \gamma_k m_{pq} Q^p_s A^q_j - \gamma_t m_{pk} Q^p_s Q^t_j)$$

$$(1.6)\qquad {}'\Gamma^i_{jk} = {}'\overset{m}{\Gamma}{}^i_{jk} + \frac{1}{2} g^{st}(\gamma_q m_{jk} Q^q_s Q^i_t - \gamma_k m_{pj} Q^p_s Q^i_t - \gamma_j m_{pk} A^p_s Q^i_t),$$

where

$$(1.7)\qquad {}''\overset{m}{\Gamma}{}^i_{jk} = \left\{{}^{\,i}_{j\,k}\right\} + \overset{\circ}{\nabla}_{[a} P^i_{k]} Q^a_j - \overset{\circ}{\nabla}_{[a} P^\ell_{k]} Q^{ai} g_{jl} - \overset{\circ}{\nabla}_{[a} P^l_{q]} Q^{ai} P_{kl} Q^q_j$$

is the metric connection, i.e. the connection with respect to which

$$D_k g_{ij} = 0,$$

while $\overset{\circ}{\nabla}_k$ denotes the ordinary covariant derivative with respect to $\left\{{}^{\,i}_{j\,k}\right\}$,

$$Q^{ai} = Q^a_s g^{si} = Q^i_s g^{sa} = Q^{ia},$$

and

$$(1.8)\qquad {}'\overset{m}{\Gamma}{}^i_{jk} = \left\{{}^{\,i}_{j\,k}\right\} + \overset{\circ}{\nabla}_{(k} P^a_{j)} Q^i_a - \overset{\circ}{\nabla}_{[a} P^l_{k]} Q^{at} Q^i_t P_{jl} - \overset{\circ}{\nabla}_{[a} P^l_{j]} Q^{at} Q^i_t P_{kl}.$$

Connection (1.8) is not a metric connection; it is only the connection satisfying, together with (1.7), condition (1.1).

We say that space satisfying conditions a´), b´) and c) is a Weyl-Otsuki space of the second kind if $m_{ij} = P_{ij}$. In the case $m_{ij} = P_{ia} P^a_j$, we say that the considered space is a Weyl-Otsuki space of the third kind.

In section 2 we investigate $W\text{-}O_n$ spaces of the second kind, and in section 3 - $W\text{-}O_n$ spaces of the third kind. In this last section (i.e. in section 3), we generalize some other results obtained in |4|.

2. WEYL-OTSUKI SPACES OF THE SECOND KIND

In this case $m_{ij} = g_{ia}P^a_j$, and connection (1.5) has the form

$$"\Gamma^i_{jk} = "\overset{m}{\Gamma}{}^i_{jk} + H^i_{jk}$$

where

$$H^i_{jk} = -\frac{1}{2}(\gamma_k Q^i_j + \delta^i_k \tilde{\gamma}_j - \tilde{\gamma}^i g_{kj}), \quad \tilde{\gamma}_j = \gamma_a Q^a_j, \quad \tilde{\gamma}^i = \tilde{\gamma}_a g^{ai}. \tag{2.1}$$

In this section we denote: by $\overset{m}{\nabla}$ the ordinary covariant derivative with rescpect to the metric connection $"\overset{m}{\Gamma}{}^i_{jk}$ (i.e. with respect to connection (1.7), by $"R$ the curvature tensor of the connection $"\Gamma$ and by $\overset{m}{R}$ - the curvature tensor of the metric connection $"\overset{m}{\Gamma}$ i.e.:

$$"R^i_{rkj} = \frac{\partial}{\partial x^k}\, "\Gamma^i_{rj} - \frac{\partial}{x^j}\, "\Gamma^i_{rk} + "\Gamma^s_{rj}\, "\Gamma^i_{sk} - "\Gamma^s_{rk}\, "\Gamma^i_{sj},$$

$$\overset{m}{R}{}^i_{rkj} = \frac{\partial}{\partial x^k}\, "\overset{m}{\Gamma}{}^i_{rj} - \frac{\partial}{x^j}\, "\overset{m}{\Gamma}{}^i_{rk} + "\overset{m}{\Gamma}{}^s_{rj}\, "\overset{m}{\Gamma}{}^i_{sk} - "\overset{m}{\Gamma}{}^s_{rk}\, "\overset{m}{\Gamma}{}^i_{sj}.$$

It is ease to see that

$$"R^i_{rkj} = \overset{m}{R}{}^i_{rkj} + \overset{m}{\nabla}_k H^i_{rj} - \overset{m}{\nabla}_j H^i_{rk} + H^s_{rj} H^i_{sk} - H^s_{rk} H^i_{sj}.$$

Taking into account (2.1) and the fact that $"\overset{m}{\Gamma}$ is a metric connection, we obtain

$$\begin{aligned}
"R_{irkj} = {} & \overset{m}{R}_{irkj} + \frac{1}{2}(\overset{m}{\nabla}_j\gamma_k - \overset{m}{\nabla}_k\gamma_j)Q_{ir} + \frac{1}{2}\gamma_k\overset{m}{\nabla}_j Q_{ir} - \frac{1}{2}\gamma_j\overset{m}{\nabla}_k Q_{ir} \\
& - \frac{1}{2} g_{ij}\overset{m}{\nabla}_k\tilde{\gamma}_r + \frac{1}{2} g_{ik}\overset{m}{\nabla}_j\tilde{\gamma}_r + \frac{1}{2} g_{jr}\overset{m}{\nabla}_k\tilde{\gamma}_i - \frac{1}{2} g_{kr}\overset{m}{\nabla}_j\tilde{\gamma}_i \\
& + \frac{1}{4}(\gamma_j\tilde{\gamma}_p Q^p_r g_{ik} - \gamma_j\tilde{\gamma}_i Q_{rk} + \gamma_k\tilde{\gamma}_r Q_{ij} - \gamma_k\tilde{\gamma}^p Q_{ip} g_{ir} \\
& - \gamma_k\tilde{\gamma}_p Q^p_r g_{ij} + \gamma_k\tilde{\gamma}_i Q_{rj} - \gamma_j\tilde{\gamma}_r Q_{ik} + \gamma_j\tilde{\gamma}^p Q_{ip} g_{kr}) \\
& + \frac{1}{4}(\tilde{\gamma}_r\tilde{\gamma}_j g_{ik} - g_{jr} g_{ik}\tilde{\gamma}_p\tilde{\gamma}^p + \tilde{\gamma}_i\tilde{\gamma}_k g_{jr} \\
& - \tilde{\gamma}_r\tilde{\gamma}_k g_{ij} + g_{kr} g_{ij}\tilde{\gamma}_p\tilde{\gamma}^p - \tilde{\gamma}_i\tilde{\gamma}_j g_{kr}).
\end{aligned} \tag{2.2}$$

Interchanging in (2.2) the place of the indices i and r and that for the indices k and j and adding the obtained relation to (2.2), we get

$$(2.3)\qquad {}''R_{irkj} + {}''R_{rijk} = \overset{m}{R}_{irkj} + \overset{m}{R}_{rijk} - g_{ij}\Theta_{kr} + g_{ik}\Theta_{jr} + g_{jr}\Theta_{ki} - g_{kr}\Theta_{ji} ,$$

where

$$\Theta_{kr} = \overset{m}{\nabla}_k\gamma_r + \tilde{\gamma}_r\tilde{\gamma}_k - \frac{1}{2} g_{kr}\tilde{\gamma}_p\tilde{\gamma}^p .$$

Introducing the notations

$$''R_{rk} = g^{ij}({}''R_{irkj} + {}''R_{rijk}), \quad \overset{m}{R}_{rk} = g^{ij}(\overset{m}{R}_{irkj} + \overset{m}{R}_{rijk})$$

$$''R = g^{rk}\ ''R_{rk} , \qquad \overset{m}{R} = g^{rk}\overset{m}{R}_{rk} ,$$

and transvecting (2.3) with g^{ij}, we find

$$(2.4)\qquad ''R_{rk} = \overset{m}{R}_{rk} + (2-n)\Theta_{kr} - g_{kr}\Theta_{ji}g^{ji} .$$

Transvecting (2.4) with g^{rk}, we obtain

$$\Theta_{ab}g^{ab} = \frac{1}{2(1-n)} ({}''R - \overset{m}{R}) .$$

Substituting this into (2.4), we get

$$(2.5)\qquad \Theta_{kr} = \frac{1}{2-n} ({}''R_{rk} - \overset{m}{R}_{rk}) + \frac{g_{rk}}{2(1-n)(2-n)} ({}''R - \overset{m}{R}) .$$

Taking into account (2.5), we express (2.3) as follows:

$$(2.6)\qquad \begin{aligned} &''R_{irkj} + {}''R_{rijk} + \frac{1}{2-n}(g_{ij}{}''R_{rk} - g_{ik}{}''R_{rj} - g_{jr}{}''R_{ik} + g_{kr}{}''R_{ij}) \\ &+ \frac{''R}{(n-1)(n-2)} (g_{ij}g_{kr} - g_{ik}g_{rj}) = \\ &= \overset{m}{R}_{irkj} + \overset{m}{R}_{rijk} + \frac{1}{2-n} (g_{ij}\overset{m}{R}_{rk} - g_{ik}\overset{m}{R}_{rj} - g_{jr}\overset{m}{R}_{ik} + g_{kr}\overset{m}{R}_{ij}) \\ &+ \frac{\overset{m}{R}}{(n-1)(n-2)} (g_{ij}g_{kr} - g_{ik}g_{rj}) . \end{aligned}$$

The tensor on the right-hand side of (2.6) does not depend on the vector γ_i. Thus we have

THEOREM 1. *The tensor*

$$"R_{irkj} + "R_{rijk} + \frac{1}{n-2}(g_{ik}"R_{rj} - g_{ij}"R_{rk} + g_{jr}"R_{ik} - g_{kr}"R_{ij})$$
$$+ \frac{"R}{(n-1)(n-2)}(g_{ij}g_{kr} - g_{ik}g_{rj})$$

does not depend on the vector field γ_i *, i.e. it is the same for all* $W-O_n$ *spaces of the second kind.*

3. WEYL - OTSUKI SPACES OF THE THIRD KIND

In this case $m_{ij} = P_{ia}P^a_j$, and connection (1.5) has the form:

$$"\Gamma^i_{jk} = "\overset{m}{\Gamma}{}^i_{jk} + \frac{1}{2}(\gamma_k\delta^i_j + P^i_k\tilde{\gamma}_j - P_{jk}\tilde{\gamma}^i) \tag{3.1}$$

where

$$\tilde{\gamma}_i = \gamma_a Q^a_i \quad , \quad \tilde{\gamma}^i = \tilde{\gamma}_a g^{ai} \quad ,$$

while connection (1.6) has the form

$$'\Gamma^i_{jk} = '\overset{m}{\Gamma}{}^i_{jk} - \frac{1}{2}(\gamma_k\delta^i_j + \gamma_j\delta^i_k - \gamma_q Q^q_s Q^{is} P_{ja} P^a_k) \ . \tag{3.2}$$

Let the metric tensor g_{ij} now undergoe the conformal transformation

$$\bar{g}_{ij} = e^{2\sigma}g_{ij} \quad , \quad \bar{g}^{ij} = e^{-2\sigma}g^{ij} \ .$$

Then the Christoffel symbols formed with respect to the two tensors are related as follows

$$\overline{\{{}^{\,i}_{j\,k}\}} = \{{}^{\,i}_{j\,k}\} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - g_{jk}\sigma^i \quad , \quad \sigma_i = \frac{\partial\sigma}{\partial x^i}, \sigma^i = g^{ia}\sigma_a$$

Obviously the basic tensor P and the basic vector γ are invariant under conformal transformationm because these are independent from g_{ij}, i.e.

$$\bar{P}^i_j = P^i_j \ , \quad \bar{Q}^i_j = Q^i_j \ , \quad \bar{\gamma}_i = \gamma_i$$

Then $''\bar{\Gamma}^i_{jk}$ and $'\bar{\Gamma}^i_{jk}$ can be expressed in the form

$$(3.3) \qquad ''\bar{\Gamma}^i_{jk} = ''\overset{m}{\bar{\Gamma}}{}^i_{jk} + \frac{1}{2}\,(\gamma_k \delta^i_j + P^i_k \tilde{\gamma}_j - P_{jk}\tilde{\gamma}^i) \ ,$$

$$(3.4) \qquad '\bar{\Gamma}^i_{jk} = '\overset{m}{\bar{\Gamma}}{}^i_{jk} - \frac{1}{2}\,(\gamma_k \delta^i_j + \gamma_j \delta^i_k - \gamma_q Q^q_s Q^{is} P_{ja} P^a_k).$$

Denoting by $\overset{o}{\bar{\nabla}}$ the ordinary covariant derivative with respact to $\{{}^{\,i}_{j\,k}\}$, and taking into account that

$$\overset{o}{\bar{\nabla}}_k P^i_j = \overset{o}{\nabla}_k P^i_j + \delta^i_k \sigma_s P^s_j - P_{jk}\sigma^i - P^i_k \sigma_j + g_{jk} P^i_s \sigma^s \ ,$$

we easely find

$$''\overset{m}{\bar{\Gamma}}{}^i_{jk} = ''\overset{m}{\Gamma}{}^i_{jk} + \delta^i_j \sigma_k + P^i_k \sigma_a Q^a_j - P_{kj}\sigma_a Q^{ai} \ ,$$

$$'\overset{m}{\bar{\Gamma}}{}^i_{jk} = '\overset{m}{\Gamma}{}^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma_q Q^q_s Q^{is} P_{ja} P^a_k \ .$$

Substituting this into (3.3) respective (3.4), we get

$$(3.5) \qquad ''\bar{\Gamma}^i_{jk} = ''\overset{m}{\Gamma}{}^i_{jk} + \delta^i_j(\sigma_k + \tfrac{1}{2}\gamma_k) + P^i_k(\sigma_a + \tfrac{1}{2}\gamma_a)Q^a_j - P_{jk}(\sigma_a + \tfrac{1}{2}\gamma_a)Q^{ai} \ ,$$

$$(3.6) \qquad '\bar{\Gamma}^i_{jk} = '\overset{m}{\Gamma}{}^i_{jk} + \delta^i_j(\sigma_k - \tfrac{1}{2}\gamma_k) + \delta^i_k(\sigma_j - \tfrac{1}{2}\gamma_j) - P_{ja}P^a_k Q^q_s Q^{is}(\sigma_q - \tfrac{1}{2}\gamma_q) \ .$$

Comparing (3.1) with (3.5), and (3.2) with (3.6), we see that:

Under conformal transformation, each of the connections $'\Gamma$, $''\Gamma$ of a **W** *-O_n -space of the third kind transforms into the connection of the same form.*

We suppose now that one of the conditions

$$(A) \quad \overset{o}{\nabla}_k P^i_j = \pi_k P^i_j \ ;$$

$$(B) \quad \overset{o}{\nabla}_k P_{ij} = \pi_i P_{kj} + \pi_j P_{ki} \quad \text{(or, equivalently, } \overset{o}{\nabla}_k P^i_j = \pi^i P_{jk} + \pi_j P^i_k)$$

is satisfied.

First, we investigate connection (3.5). Taking into account (1.7), it is easy to see that

$$"\overset{m}{\Gamma}{}^{i}_{jk} = \{{}_{j}{}^{i}{}_{k}\} + \varepsilon\pi_a Q^q_j P^i_k - \varepsilon\pi_a Q^{ai} P_{kj} \ ,$$

where $\varepsilon = +1$ if condition (A) is satisfied, and $\varepsilon = -1$, if condition (B) is satisfied. Substituting this into (3.5), we obtain

$$"\bar{\Gamma}^{i}_{jk} = \{{}_{j}{}^{i}{}_{k}\} + \delta^i_j(\sigma_k + \tfrac{1}{2}\gamma_k) + P^i_k(\sigma_a + \tfrac{1}{2}\gamma_a + \varepsilon\pi_a)Q^a_j$$

$$(3.7) \qquad - P_{jk}(\sigma_a + \tfrac{1}{2}\gamma_a + \varepsilon\pi_a)Q^{ai} \ ,$$

If we put

$$\sigma_k + \tfrac{1}{2}\gamma_k = S_k \ , \qquad (\sigma_a + \tfrac{1}{2}\gamma_a + \varepsilon\pi_a)Q^a_j = \tilde{S}_j \ ,$$

we may express (3.7) in the form

$$"\bar{\Gamma}^{i}_{jk} = \{{}_{j}{}^{i}{}_{k}\} + \delta^i_j S_k + P^i_k \tilde{S}_j - P_{kj}\tilde{S}^i \ .$$

Let us denote by $"\bar{R}^i_{rkj}$ the curvature tensor of connection $"\bar{\Gamma}$, and by K^i_{rkj} - the curvature tensor of connection $\{{}_{j}{}^{i}{}_{k}\}$. Then we have

$$"\bar{R}_{irkj} = K_{irkj} + g_{ir}(\overset{o}{\nabla}_k S_j - \overset{o}{\nabla}_j S_k)$$

$$+ P_{ij}(\overset{o}{\nabla}_k\tilde{S}_r + \varepsilon\tilde{S}_r\pi_k - \tilde{S}_r P^p_k S_p + \tfrac{1}{2}P_{kr}\tilde{S}^p\tilde{S}_p)$$

$$(3.8) \qquad - P_{ik}(\overset{o}{\nabla}_j\tilde{S}_r + \varepsilon\tilde{S}_r\pi_j - \tilde{S}_r P^p_j\tilde{S}_p + \tfrac{1}{2}P_{jr}\tilde{S}^p\tilde{S}_p)$$

$$- P_{jr}(\overset{o}{\nabla}_k\tilde{S}_i + \varepsilon\tilde{S}_i\pi_k - \tilde{S}_i P^p_k\tilde{S}_p + \tfrac{1}{2}P_{ik}\tilde{S}^p\tilde{S}_p)$$

$$+ P_{kr}(\overset{o}{\nabla}_j\tilde{S}_i + \varepsilon\tilde{S}_i\pi_j - \tilde{S}_i P^p_j\tilde{S}_p + \tfrac{1}{2}P_{ij}\tilde{S}^p\tilde{S}_p) \ .$$

If we interchange the indices i and r and the indices k and j and add the obtained relation to (3.8), we get:

$$\tfrac{1}{2}("\bar{R}_{arkj} + "\bar{R}_{rajk}) = K_{arkj} + P_{rk}\Psi_{ja} - P_{rj}\Psi_{ka} - P_{ka}\Psi_{jr} + P_{ja}\Psi_{kr} \ ,$$

or, transvecting with g^{ia}

$$\frac{1}{2}({}''\bar{R}_{arkj} + {}''\bar{R}_{rajk})g^{ia} = \tag{3.9}$$
$$= K^{i}_{rkj} + P_{rk}\Psi^{i}_{j} - P_{rj}\Psi^{i}_{k} - P^{i}_{k}\Psi_{jr} + P^{i}_{j}\Psi_{kr} ,$$

where

$$\Psi_{ji} = \overset{o}{\nabla}_j \tilde{S}_i + \varepsilon\tilde{S}_i\pi_j - \tilde{S}_i P^{p}_{j}\tilde{S}_p + \frac{1}{2} P_{ji}\tilde{S}^p\tilde{S}_p , \qquad \Psi^{i}_{j} = \Psi_{ja}g^{ai} .$$

Introducing the notations

$$''\bar{R}^{*}_{kr} = \frac{1}{2}({}''\bar{R}_{arkb} + {}''\bar{R}_{rabk})Q^{ab} , \qquad ''\bar{R}^{*r}_{k} = {}''\bar{R}^{*}_{ka}\, g^{ar} ,$$

$$\overset{*}{K}_{kr} = K^{a}_{rkb}Q^{b}_{a} , \qquad \overset{*}{K}{}^{r}_{k} = \overset{*}{K}_{ka}g^{ar} ,$$

$$''\bar{R}^{*} = {}''\bar{R}^{*}_{kr}Q^{kr} , \qquad \overset{*}{K} = \overset{*}{K}_{kr}Q^{kr} ,$$

and transvecting (3.9) with Q^{j}_{i} , we find:

$$''\bar{R}^{*}_{kr} = \overset{*}{K}_{kr} + (n-2)\Psi_{kr} + P_{kr}\Psi_{ji}Q^{ji} . \tag{3.10}$$

Transvecting (3.10) with Q^{rk} , we have

$$\Psi_{ji}Q^{ji} = \frac{1}{2(n-1)} ({}''\bar{R}^{*} - \overset{*}{K}).$$

Substituting this into (3.10) , we get

$$\Psi_{kr} = \frac{1}{n-2} ({}''\bar{R}^{*}_{kr} - \overset{*}{K}_{kr}) - \frac{P_{kr}}{2(n-1)(n-2)} ({}''\bar{R}^{*} - \overset{*}{K}).$$

Finally, inserting this into (3.9), we obtain:

$$\frac{1}{2}({}''\bar{R}_{arkj} + {}''\bar{R}_{rajk})g^{ia} + \frac{1}{n-2} (P_{rj}{}''\bar{R}^{*i}_{k} - P_{rk}{}''\bar{R}^{*i}_{j} + P^{i}_{k}{}''\bar{R}^{*}_{jr} - \tag{3.11}$$
$$- P^{i}_{j}{}''\bar{R}^{*}_{kr}) - \frac{''\bar{R}^{*}}{(n-1)(n-2)}(P_{rj} P^{i}_{k} - P_{rk} P^{i}_{j}) =$$
$$= K^{i}_{rkj} + \frac{1}{n-2} (P_{rj}\overset{*}{K}{}^{i}_{k} - P_{rk}\overset{*}{K}{}^{i}_{j} + P^{i}_{k} \overset{*}{K}_{jr} - P^{i}_{j}\overset{*}{K}_{kr})$$
$$- \frac{\overset{*}{K}}{(n-1)(n-2)} (P_{rj} P^{i}_{k} - P_{rk} P^{i}_{j}).$$

The tensor on the right-hand side of (3.11) depends only on the basic tensor P^i_j and g_{ij}. Therefore, we have

THEOREM 2. *If condition (A) or condition (B) is satisfied, the tensor on the left-hand side of (3.11) is invariant with respect to the conformal transformation. This tensor does not depend on the vectors* π_i *and* γ_i *as well.*

We are to investingate connection (3.6). Taking into account (1.8) and condition (A), we find

$$\acute{\bar{\Gamma}}^i_{jk} = \{^{\ i}_{j\ k}\} + \delta^i_j(\sigma_k - \tfrac{1}{2}\gamma_k + \pi_k) + \delta^i_k(\sigma_j - \tfrac{1}{2}\gamma_j + \pi_j) - $$
$$- P_{ja}P^a_k Q^q_s Q^{is}(\sigma_q - \tfrac{1}{2}\gamma_q + \pi_q) .$$

Putting

$$\sigma_k - \frac{1}{2}\gamma_k + \pi_k = V_k ,$$

we may re-write this connection as follows:

$$\acute{\bar{\Gamma}}^i_{jk} = \{^{\ i}_{j\ k}\} + \delta^i_j V_k + \delta^i_k V_j - P_{ja}P^a_k Q^q_s Q^{is} V_q . \tag{3.12}$$

Let us denote by $\acute{\bar{R}}^i_{rkj}$ the curvature tensor of the connection $\acute{\bar{\Gamma}}$. Then we have

$$\begin{aligned}
\acute{\bar{R}}^i_{rkj} = {} & K^i_{rkj} + \delta^i_r(\overset{o}{\nabla}_k V_j - \overset{o}{\nabla}_j V_k) \\
& + \delta^i_j(\overset{o}{\nabla}_k V_r - V_r V_k + \tfrac{1}{2}V_l V_p Q^l_s Q^{ps} P_{ra}P^a_k) \\
& - \delta^i_k(\overset{o}{\nabla}_j V_r - V_r V_j + \tfrac{1}{2} V_l V_p Q^l_s Q^{ps} P_{ra}P^a_j) \\
& + P_{ra}P^a_k(\overset{o}{\nabla}_j V_l - V_j V_l + \tfrac{1}{2}V_m V_p Q^m_s Q^{ps} P_{la}P^a_j) Q^{it} Q^l_t \\
& - P_{ra}P^a_j(\overset{o}{\nabla}_k V_l - V_k V_l + \tfrac{1}{2}V_m V_p Q^m_s Q^{ps} P_{la}P^a_k) Q^{it} Q^l_t
\end{aligned} \tag{3.13}$$

because of

$$\overset{o}{\nabla}_k(Q^l_s Q^i_t P^a_r P^b_j) = \overset{o}{\nabla}_j(Q^l_s Q^i_t P^a_r P^b_k) = 0$$

and

$$P_{la}P^a_j Q^l_s Q^{is} = \delta^i_j , \quad P_{la}P^a_k Q^{is} Q^l_s = \delta^i_k .$$

Contracting with respect to i and r, we get

$$\overset{o}{\nabla}_k V_j - \overset{o}{\nabla}_j V_k = \frac{1}{n}\, {}'\bar{R}^a{}_{akj} .$$

Substituting this into (3.13) and putting

$$\phi_{kr} = \overset{o}{\nabla}_k V_r - V_k V_r + \frac{1}{2} V_l V_p Q^l_s Q^{ps} P_{ra} P^a_k ,$$

we have

$${}'\bar{R}^i_{rkj} - \frac{1}{n}\delta^i_r\, {}'\bar{R}^r_{akj} = \mathcal{K}^i_{rkj} + \delta^i_j \phi_{kr} - \delta^i_k \phi_{jr}$$

$$(3.14) \qquad + P_{ra}P^a_k \phi_{jl} Q^{is} Q^l_s - P_{ra}P^a_j \phi_{kl} Q^{is} Q^l_s .$$

Introducting the notation

$${}'\bar{R}_{rk} = {}'\bar{R}^a_{rka} - \frac{1}{n}\, {}'\bar{R}^a_{akr} ,$$

and contracting (3.14) for i and j, we obtain

$${}'\bar{R}_{rk} = \mathcal{K}_{rk} + (n-2)\phi_{kr} + P_{rt}P^t_k Q^b_a Q^p_b \phi^a_p .$$

Transvecting this with $Q^{pr}Q^k_p$, we find

$$Q^{ba}Q^p_b \phi_{pa} = \frac{1}{2(n-1)} \left({}'\bar{R}_{rk} Q^{pr} Q^k_p - \mathcal{K}_{rk} Q^{pr} Q^k_p \right) .$$

Thus we have

$$\phi_{kr} = \frac{1}{n-2} \left({}'\bar{R}_{rk} - \mathcal{K}_{rk} \right) - \frac{P_{rt}P^t_k}{2(n-1)(n-2)} \left({}'\bar{R}_{ab} Q^{pa} Q^b_p - \mathcal{K}_{ab} Q^{pa} Q^b_p \right) .$$

Substituting this into (3.14), we obtain

$$
\begin{aligned}
&{}'\bar{R}^i_{rkj} - \frac{1}{n}\delta^i_r\, {}'\bar{R}^a_{akj} - \frac{1}{n-2}\big(\delta^i_j\, {}'\bar{R}_{rk} - \delta^i_k\, {}'\bar{R}_{rj} - P_{rt}P^t_j Q^{ba} Q^i_b\, {}'\bar{R}_{ak} \\
&\qquad + P_{tr}P^t_k Q^{ba} Q^i_b\, {}'\bar{R}_{aj}\big) \\
&\quad + \frac{{}'\bar{R}_{ab} Q^{pa} Q^b_p}{(n-1)(n-2)}\big(\delta^i_j P_{rt}P^t_k - \delta^i_k P_{rt}P^t_j\big) = \\
&\quad = K^i_{rkj} - \frac{1}{n-2}\big(\delta^i_j K_{rk} - \delta^i_k K_{rj} - P_{rt}P^t_j Q^{ba} Q^i_b K_{ak} \\
&\qquad + P_{rt}P^t_k Q^{ba} Q^i_b K_{aj}\big) + \\
&\quad + \frac{K_{ab}\, Q^{pa} Q^b_a}{(n-1)(n-2)}\big(\delta^i_j P_{rt}P^t_k - \delta^i_k P_{rt}P^t_j\big) .
\end{aligned} \tag{3.15}
$$

In case condition (B) is satisfied, we start with (3.6), which, putting

$$\sigma_k - \frac{1}{2}\gamma_k = T_k ,$$

can be re-written in the form

$$ {}'\bar{\Gamma}^i_{jk} = {}'\overset{m}{\Gamma}{}^i_{jk} + T_k\delta^i_j + T_j\delta^i_k - T_a Q^a_s Q^{is} P_{jb}P^b_k . \tag{3.16}$$

This form is the same as (3.12). Only, in (3.12) the connection $\{ {}_j{}^i{}_k \}$ depends of the basic tensor g_{ij}, while here connection ${}'\overset{m}{\Gamma}{}^i_{jk}$ depends of the bacic tensor g_{ij} and P^i_j (see (1.8)). Besides, denoting by $\overset{m}{\nabla}$ the ordinary covariant derivative with respect tp ${}'\overset{m}{\Gamma}{}^i_{jk}$, and using condition (B), we find:

$$
\begin{aligned}
&{}'\overset{m}{\Gamma}{}^i_{jk} = \{ {}_j{}^i{}_k \} + \pi^a Q^i_a P_{kj} + \pi_a Q^{at} Q^i_t P^b_k P_{jb} ; \\
&\overset{o}{\nabla}_k Q^i_j = -\pi^a Q^i_a g_{kj} - \pi_a Q^a_j \delta^i_k ; \\
&(\overset{m}{\nabla}_k Q^l_s) Q^{is} = -\pi_a Q^a_s Q^{is}\delta^l_k - \pi_a Q^{at} Q^h_t Q^l_h P^i_k , \\
&Q^l_s(\overset{m}{\nabla}_k Q^{is}) = \pi_a Q^a_s Q^{is}\delta^l_k + \pi_a Q^{at} Q^s_r Q^l_s P^i_k ,
\end{aligned} \tag{3.17}
$$

i.e.

$$\overset{m}{\nabla}_k (\Omega^l_s Q^{is}) = 0;$$

similary

$$\overset{m}{\nabla}_k (P_{ra} P^a_j) = 0 .$$

Thus, proceeding with connection (3.16) in the same manner as with (3.12), we find instead of (3.15), the relation

$$\begin{aligned}
{}'\bar{R}^i_{rkj} - \frac{1}{n}\,\delta^i_r\,{}'\bar{R}^a_{akj} - \frac{1}{n-2}\,(\delta^i_j\,{}'\bar{R}_{rk} - \delta^i_k\,{}'\bar{R}_{rj} - P_{rt}P^t_j Q^{ba} Q^i_b\,{}'\bar{R}_{ak} \\
+ P_{rt}P^t_k Q^{ba} Q^i_b\,{}'\bar{R}_{aj}) \\
+ \frac{{}'\bar{R}_{ab} Q^{pa} Q^b_p}{(n-1)(n-2)}\,(\delta^i_j P_{rt} P^t_k - \delta^i_k P_{rt} P^t_j) = \\
= \overset{m}{K}{}^i_{rkj} - \frac{1}{n}\,\delta^i_r \overset{m}{K}{}^a_{akj} - \frac{1}{n-2}(\delta^i_j \overset{m}{K}_{rk} - \delta^i_k \overset{m}{K}_{rj} - P_{rt}P^t_j Q^{ba} Q^i_b \overset{m}{K}_{ak} \\
+ P_{rt}P^t_k Q^{ba} Q^i_b \overset{m}{K}_{aj}) \\
+ \frac{\overset{m}{K}_{ab} Q^{pa} Q^b_p}{(n-1)(n-2)}\,(\delta^i_j P_{rt} P^t_k - \delta^i_k P_{rt} P^t_j) ,
\end{aligned}$$

where $\overset{m}{K}{}^i_{rkj}$ is the curvature tensor with respect to connection (3.17) and $\overset{m}{K}_{rk} = \overset{m}{K}{}^a_{rka}$.

Therefore, we have

THEOREM 3. *If condition (A) or condition (B) is satisfied, besides the tensor on the left-hand side of (3.11), the tensor*

$$\begin{aligned}
{}'\bar{R}^i_{rkj} - \frac{1}{n}\,\delta^i_r\;{}'\bar{R}^a_{akj} \\
- \frac{1}{n-2}(\delta^i_j\,{}'\bar{R}_{rk} - \delta^i_k\,{}'\bar{R}_{rj} - P_{rt}P^t_j Q^{ba} Q^i_b\,{}'\bar{R}_{ak} + P_{rt}P^t_k Q^{ba} Q^i_b\;{}'\bar{R}_{aj}) \\
+ \frac{{}'\bar{R}_{ab} Q^{pa} Q^b_p}{(n-1)(n-2)}\,(\delta^i_j P_{rt} P^t_k - \delta^i_k P_{rt} P^t_j)
\end{aligned}$$

is invariant with respect to the conformal transformation, too. This tensor does not depend on the vector field γ_i *and if condition (A) is satisfied it does not depend on the vector field* π_i *either.*

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REZIME

WEYL-OTSUKI-JEVI PROSTORI DRUGE I TREĆE VRSTE

U ovom radu ispituje se ona opšta regularna koneksija Otsuki-jevog prostora koja zadovoljava uslove a') i c) i čiji je kontravarijantni deo $'\Gamma$ simetričan. Ta je koneksija oblika (1.5),(1.6),(1.7) i (1.8). Ako je, pri tom, $m_{ij}=P_{ij}$, odnosno $m_{ij}=P_p P^a_j$, posmatrani prostor je Weyl-Otsuki-jev prostor ($W-O_n$-prostor) druge odnosno treće vrste.

U §2 je dokazano da je tenzor (2.6) zajednički za sve $W-O_n$ -prostore druge vrste.

U §3 ispituju se konformne transformacije $W-O_n$ -prostora treće vrste. Dokažana je teorema:

Ako je zadovoljen uslov (A) ili uslov (B), tenzor (3.11) je invarijantan u odnosu na konformne transformacije. Taj tenzor ne zavisi ni od polja vektora π_i i γ_i .