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ON THE ORTHOGONAL SPACES OF THE SUBSPACES OF A RIEMANN - OTSUKI SPACE

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INTRODUCTION

In this paper we suppose that a Riemann-Otsuki space $R\text{-}O_n$ (see |1|) with the symmetric metrix tensor satisfying the relation

(1) $$Dg_{ij} = 0 \quad (\det(g_{ij}) \neq 0)$$

and the P^i_j $(\det(P^i_j) \neq 0)$ as basic objects is given. By our investigation g_{ij} and P_j satisfying throughout the relation $P^i_j g_{is} = P^i_s g_{ij}$.

The Otsuki's covariant differential of a tensor T^i_j is defined by

(2) $$DT^i_j := \nabla_k T^i_j dx^k := P^i_a P^b_j (\partial_k T^a_b + {}'\Gamma_s{}^a{}_k T^s_b - {}''\Gamma_b{}^s{}_k T^a_s) dx^k$$

where the coefficients of the connections $'\Gamma$ and $''\Gamma$ and the tensor P^i_j satisfy the Otsuki's relation (see |2| (3.13))

(3) $$\partial_k P^s_r + P^i_r {}''\Gamma_i{}^s{}_r - {}'\Gamma_r{}^i{}_k P^s_i = 0.$$

Let as usual by $x^i = x^i(u^1, \ldots, u^m)$ $(m < n)$ be defined an m-dimensional subspace S_m. We suppose that the rank $\|\partial x^i/\partial n^\alpha\| = m$/* and use the notation $\xi^i_\alpha := \partial x^i/\partial u^\alpha$. Let the metric tensor

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/* In this paper Latin indices run from 1 to n, Greek indices $\alpha, \beta, \ldots, \lambda$ run from 1 to m, but $\mu, \nu, \ldots, \omega$ run from (m+1) to n. In the following the index runing from (m+1) to n will be co-or contravariant so as the original index was.

of S_m be the projection of g_{ij}, that is

(4) $$G_{\alpha\beta} := g_{ij}\xi^i_\alpha \xi^j_\beta \ .$$

We define, us usual, the contravariant components of $G_{\alpha\beta}$ by $G^{\alpha\beta}$ i.e. $G_{\alpha\beta}G^{\beta\gamma} = \delta^\gamma_\alpha$ and the contravariant components of the projection vectors by

(5) $$\xi^\alpha_a := g_{ab}G^{\alpha\beta}\xi^b_\beta \ .$$

Obviously we have $G^{\alpha\beta}=g^{ab}\xi^\alpha_a\xi^\beta_b$, where we use for the tangent and normal vectors the relations

(6) $$\xi^i_{\alpha}\underset{\mu}{N}_i = 0; \quad \underset{\mu}{N}_i \underset{\nu}{N}^i = \delta^\mu_\nu ; \quad \underset{\mu}{N}_j = g_{ij}\underset{\mu}{N}^i ; \ \xi^i_\alpha \xi^\alpha_j + \underset{\mu}{N}^i \underset{\mu}{N}_j = \delta^i_j$$

where $\underset{\mu}{N}_i$ are mutually orthogonal unit vectors (see |2| and |3|). This relations are very useful.

Let the P-tensor of S_m be the projection rensor P^α_β defined by

(7) $$P^\alpha_\beta := P^i_j \xi^j_\beta \xi^\alpha_i \ .$$

The covariant differential over S_m of a tensor T^α_β of S_m we define by

(8) $$\overset{*}{D}T^\alpha_\beta := \overset{*}{\nabla}_\gamma T^\alpha_\beta du^\gamma = P^\alpha_\varepsilon P^\eta_\beta (\partial_\gamma T^\varepsilon_\eta + {}'\overset{*}{\Gamma}{}^\varepsilon_{\zeta\gamma} T^\zeta_\eta - {}''\overset{*}{\Gamma}{}^\zeta_{\eta\gamma} T^\varepsilon_\zeta) du^\gamma \ .$$

In |1| it was proved that the relation

(9) $${}''\overset{*}{\Gamma}{}^\alpha_{\beta\ \gamma} := {}''\Gamma^\alpha_{\beta\ \gamma} + \xi^i_{\beta\gamma}\xi^\alpha_i \quad ({}''\Gamma^\alpha_{\beta\ \gamma} := {}''\Gamma^i_{j\ k}\xi^\alpha_i \xi^j_\beta \xi^k_\gamma ; \quad \xi^i_{\beta\gamma} := \frac{\partial}{\partial u^\gamma}\xi^i_\beta)$$

is necessary and suficient condition to be $\overset{*}{D}G_{\alpha\beta} = 0$ and it is easy to prove that the generalization

(10) $$\overset{*}{D}T_{\alpha_1 \dots \alpha_k} = \xi^{i_1}_{\alpha_1} \dots \xi^{i_k}_{\alpha_k} DT_{i_1 \dots i_k}$$

of (22) in |1| holds if $T_{i_1 \dots i_k} = \xi^{\alpha_1}_{i_1} \dots \xi^{\alpha_k}_{i_k} T_{\alpha_1 \dots \alpha_k}$, i.e. it is a tensor of T_m.

In Paragraph 1 we determine a sufficient condition, that a subspace S_m of a Riemann-Otsuki space will be an R-O_m

and consider some consequences of this condition. In Paragraph 2 we consider the spaces S_{n-m}, orthogonal to S_m, and determine the coefficients of their connection. At the end we prove that the condition considered in Paragraph 1 is sufficient for S_{n-m} to be a Riemann-Otsuki space.

1. THE BASIC CONDITIONS

In |1| it was proved, thet if the covariant differential of T^α, which is an element of T_m is defined by $\overset{*}{D}T^\alpha := \xi^\alpha_i DT^i$, then it must be

$$(1.1)\qquad {}'\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma := {}'\Gamma_\beta{}^\alpha{}_\gamma + \xi^i_{\beta\gamma}\xi^\alpha_i \quad ({}'\Gamma^\alpha_{\beta\gamma} := {}'\Gamma_j{}^i{}_k\, \xi^\alpha_i \xi^j_\beta \xi^k_\gamma).$$

Since we observe an Otsuki space, the connection-coefficients ${}'\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and ${}''\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and the tensor P^α_β must satisfy the relation analogous to (3), i.e. both sides of (27) |1| must vanish. This relation we can write in the form

$$(1.2)\qquad P^i_r \xi^\alpha_i \underset{\mu}{N}{}^r ({}'\Gamma_\beta{}^\alpha{}_\gamma + \xi^a_{\beta\gamma}) \underset{\mu}{N}{}_a - P^i_b \xi^b_\beta \underset{\mu}{N}{}^i ({}''\Gamma_\beta{}^\alpha{}_\gamma + \xi^\alpha_{r\gamma}) \underset{\mu}{N}{}^r = 0 .$$

Using the relations (5) and $P^i_j g_{is} = P^i_s g_{ij}$ we get

$$(1.3)\qquad P^i_r\, \xi^\alpha_i\, \underset{\mu}{N}{}^r = G^{\alpha\varepsilon} P^i_b \xi^b_\varepsilon \underset{\mu}{N}{}_i$$

or in the projection notation $P^\alpha_\mu = G^{\alpha\varepsilon} P^\mu_\varepsilon$. Substituting (1.3) in (1.2) we get the condition

$$P^i_b \xi^b_\varepsilon \underset{\mu}{N}{}_i \,|\, G^{\alpha\varepsilon}({}'\Gamma_\beta{}^a{}_\gamma + \xi^a_{\beta\gamma}) \underset{\mu}{N}{}_a - \delta^\varepsilon_\beta ({}''\Gamma^\alpha_{r\gamma} + \xi^\alpha_{r\gamma}) \underset{\mu}{N}{}^r | = 0.$$

In the following we suppose that

$$(1.4)\qquad P^i_b \xi^b_\varepsilon \underset{\mu}{N}{}^i = 0.$$

From (1.3) it follows that in this case $P^\mu_\alpha = P^\alpha_\mu = 0$, and it is a sufficient condition for S_m that Otsuki's relation between coefficients of connections ${}'\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and ${}''\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and the tensor P^α_β could be satisfied.

Now, we prove some consequences of (1.4). It is known that in Otsuki spaces there exists a tensor Q^i_j satisfying the relation

(1.5) $$P^i_j\, Q^j_s = \delta^i_s \ .$$

Let the projection of P^i_j in the direction of the vectors orthogonal to S_m be

(1.6) $$P^\mu_\nu := P^i_j N_{\mu i} N^j_\nu$$

Now, we suppose that there are tensors $\overset{*}{Q}{}^\alpha_\beta$ and $\tilde{Q}^\mu_\nu$ so that

(1.7) $$P^\alpha_\beta \overset{*}{Q}{}^\beta_\gamma = \delta^\alpha_\gamma \quad ; \quad P^\mu_\nu \tilde{Q}^\nu_\sigma = \delta^\mu_\sigma$$

hold.

THEOREM 1. *From (1.4) it follows that* $\overset{*}{Q}{}^\alpha_\beta = Q^i_j \xi^\alpha_i \xi^j_\beta = Q^\alpha_\beta$ *and* $\tilde{Q}^\mu_\nu = Q^i_j N_{\mu i} N^j_\nu = Q^\mu_\nu$.

P r o o f. Substituting (7) in (1.7), multiplying it by ξ^k_α , using the last relation of (6) and (1.4) we get

$$P^k_j \xi^j_\beta \overset{*}{Q}{}^\beta_\varepsilon = \xi^k_\varepsilon \ .$$

Multiplication by $Q^\ell_k \xi^\alpha_\ell$ according to $\xi^i_\alpha \xi^\beta_i = \delta^\beta_\alpha$ gives the affirmation of the theorem. The second part of the theorem follows in the same way, but we must take the vectors N^i_μ , instead of the ξ^i_α.

THEOREM 2. *From* $P^\mu_\alpha = 0$ *it follows that* $Q^\alpha_\mu = Q^\mu_\alpha = 0$.

P r o o f. According to the definition of P^μ_β and (1.4) we have $P^\mu_\beta = P^i_j N_{\mu i} \xi^j_\beta = 0$. Multiplying it by $\xi^\beta_k Q^k_s$ using (6) and (1.5) we get

$$N_{\mu s} = P^\mu_\rho N_{\rho k} Q^k_s \ .$$

Further, multiplication by Q^σ_μ and ξ^s_α gives

$$Q^k_s \xi^s_\alpha N_{\sigma k} = Q^\sigma_\alpha = 0 .$$

THEOREM 3. *From the relation (1.4) it follows that*

$$(1.8)\qquad a)\ \begin{aligned} P^i_j\xi^\alpha_i &= P^\alpha_\beta\xi^\beta_j \\ Q^i_j\xi^\alpha_i &= Q^\alpha_\beta\xi^\beta_j \end{aligned} \qquad b)\ \begin{aligned} P^i_j N_{\mu i} &= P^\mu_\nu N_{\nu j} \\ Q^i_j N_{\mu i} &= Q^\mu_\nu N_{\nu j} \end{aligned}$$

$$(1.9)\qquad M^i_j\xi^\alpha_i = M^\alpha_\beta\xi^\beta_j \qquad M^i_j N_{\mu i} = M^\mu_\nu N_{\nu j} \qquad (M^i_j = P^i_a P^a_j;\ M^\alpha_\beta = P^\alpha_\varepsilon P^\varepsilon_\beta;\ M^\mu_\nu = P^\mu_\sigma P^\sigma_\nu)$$

The multiplication of (1.4) by ξ^β_k according to (10) and (1.6) proves the statement of this theorem. Relations (1.8) and (1.9) are very useful and they will be often applied in the followings.

Relations (1.8) and (1.9) mean that they hold an eigen quality in all the space for the vectors ξ^α_i and $N_{\mu i}$. The subspace S_m is an eigen space, and its orthogonal space S_{n-m} is an eigen space too. If $m = n-1$, then relation (1.8b) is a simple eigen property. From (1.4) it follows directly that

$$P^i_j = P^\alpha_\beta \xi^i_\alpha \xi^\beta_j + P^\mu_\nu N^i_\mu N_{\nu j}$$

and according to the statement of the Theorem 2 it follows that

$$Q^i_j = Q^\alpha_\beta \xi^i_\alpha \xi^\beta_j + Q^\mu_\nu N^i_\mu N_{\nu j} .$$

One of consequences of (1.4) is that the relations (24) and (26) of 1 are equivalent, i.e. (1.1) holds.

2. THE CONNECTION OF THE ORTHOGONAL SPACES

In this paragraph we extend the definition of covariant differential $\overset{*}{D}$ on the elements orthogonal to S_m, but defined over it. The coefficients of the connection of co- and contravariant part of the orthogonal space we denote by $'\Lambda^\mu_{\nu\gamma}$ and $''\Lambda^\mu_{\nu\gamma}$ respectively. We must determine the conditions that the Otsuki's relation (3) between this coefficients of connections and the tensor P^μ_ν will be satisfied. The tensor P^μ_ν is the projection of the tensor P^i_j on the orthogonal subspace S_{n-m}. In the following we consider a covariant vector Y_i orthogonal to S_m and defined over it. It is expressible in the from

(2.1) $$Y_i = \underset{\mu}{N}_i Y_\mu .$$

Now we define the covariant differential $\overset{*}{D}Y_\mu$ as a projection of DY_i onto the (n-m)-dimensional direction orthogonal to S_m, i.e.

(2.2) $$\overset{*}{D}Y_\mu := g^{ij}\underset{\mu}{N}_j DY_i = \underset{\mu}{N}^i DY_i .$$

According to (2) and (2.1) it follows that

$$\overset{*}{D}Y_\mu = g^{ij}\underset{\mu}{N}_j P^a_i \underset{\rho}{N}_a(\partial_\gamma Y_\rho)du^\gamma - g^{ij}\underset{\mu}{N}_j P^a_i ({}''\Gamma^{\ s}_{a\ k}\underset{\rho}{N}_s \xi^k_\gamma - \partial_\gamma \underset{\rho}{N}_a) Y_\rho du^\gamma .$$

Using (6), (1.6) and the notations $\underset{\sigma}{N}_a {}''\Lambda^{\ \rho}_{\sigma\gamma} := {}''\Gamma^{\ s}_{a\ k}\underset{\rho}{N}_s \xi^k_\gamma - \partial_\gamma \underset{\rho}{N}_a$ or

(2.3) $${}''\Lambda^{\ \rho}_{\sigma\ \gamma} := ({}''\Gamma^{\ s}_{a\ k}\underset{\rho}{N}_s \xi^k_\gamma - \partial_\gamma \underset{\rho}{N}_a)\underset{\sigma}{N}^a$$

we get

(2.4) $$\overset{*}{D}Y_\mu = P^\rho_\mu (dY_\rho - {}''\Lambda^{\sigma}_{\rho\gamma} Y_\sigma du^\gamma) = \overset{*}{\nabla}_\gamma Y_\mu du^\gamma .$$

In the same way we define the covariant differential $\overset{*}{D}Y^\mu$ of a contravariant vector Y^μ which has the contravariant components in the basic R-O_n space. Let Y^i be orthogonal to S_m. Now we define:

(2.5) $$\overset{*}{D}Y^\mu := \underset{\mu}{N}_i DY^i .$$

Since Y^i is expressible in the from $Y^i = \underset{\mu}{N}^i Y^\mu$, using (2) it is not difficult to get, that if

$$\underset{\rho}{N}^a {}'\Lambda^{\ \rho}_{\nu\gamma} := {}'\Gamma^{\ a}_{s\ k}\underset{\nu}{N}^s \xi^k_\gamma + \partial_\gamma \underset{\nu}{N}^a$$

or

(2.6) $${}'\Lambda^{\ \rho}_{\nu\ \gamma} := ({}'\Gamma^{\ a}_{s\ k}\underset{\nu}{N}^s \xi^k_\gamma + \partial_\gamma \underset{\nu}{N}^a)\underset{\rho}{N}_a$$

then according to (2.6) relation (2.5) has the form

(2.7) $$\overset{*}{D}Y^\mu = P^\mu_\nu (\partial_\gamma Y^\nu + {}'\Lambda^{\ \nu}_{\rho\ \gamma} Y^\rho) du^\gamma = \overset{*}{\nabla}_\gamma Y^\mu du^\gamma .$$

Relations (2.4) and (2.7) show that in the subspace S_{n-m} it is possible to define a covariant differential, like in the Otsuki's spaces |2|.

Coefficients ${}'\Lambda^{\ \rho}_{\nu\ \gamma}$ and ${}''\Lambda^{\ \rho}_{\nu\ \gamma}$ are coefficients of connections of the space S_{n-m}. In this paper we consider an Otsuki space, and so we must determine conditions that Otsuki's relation

$$(2.8)\qquad \partial_\gamma P^\mu_\sigma + P^\nu_\sigma {}''\Lambda_\nu{}^\mu{}_\gamma - P^\mu_\nu {}'\Lambda_\sigma{}^\nu{}_\gamma = 0$$

will be satisfied. Substituting (2.3), (2.6) and (1.6) in(2.8), using the fact that ${}'\Gamma_j{}^i{}_k$, ${}''\Gamma_j{}^i{}_k$ and P^i_j satisfy (3), we get that it must be

$$(2.9)\qquad P^i_r \xi^r_\alpha N_\mu{}^i ({}'\Gamma_a{}^s{}_\gamma \xi^\alpha_s - \xi^\alpha_{a\gamma}) N^a_\sigma - P^i_j \xi^\alpha_i N^j_\sigma ({}''\Gamma_s{}^a{}_\gamma \xi^s_\alpha + \xi^a_{\alpha\gamma}) N_{\mu a} = 0.$$

According to the supposition (1.4), using (1.3), it follows that (2.9) is satisfied. It means, that the following holds:

THEOREM 4. *The assumption (1.4) is a sufficient condition that Otsuki's relation between coefficients of connection* ${}'\Lambda_\nu{}^\mu{}_\gamma$ *and* ${}''\Lambda_\nu{}^\mu{}_\gamma$ *and the tensor* P^μ_ν *will be satisfied, and so* S_{n-m} *is a Riemann-Otsuki space* .

After all we can define the covariant differential of a mixed tensor involving there kinds of indices, for instance a tensor $T^{i\alpha\mu}_{j\beta\nu}$. Now it is

$$(2.10)\qquad \overset{*}{D}T^{i\alpha\mu}_{j\beta\nu} := P^i_a P^b_j P^\alpha_\varepsilon P^\eta_\beta P^\mu_\rho P^\sigma_\nu (\partial_\gamma T^{a\varepsilon\rho}_{b\eta\sigma} + {}'\Gamma_s{}^a{}_\gamma T^{s\varepsilon\rho}_{b\eta\sigma} + {}'\overset{*}{\Gamma}_\chi{}^\varepsilon{}_\gamma T^{a\chi\rho}_{b\eta\sigma} + {}'\Lambda_\tau{}^\rho{}_\gamma T^{a\varepsilon\tau}_{b\eta\sigma} - {}''\Gamma_b{}^s{}_\gamma T^{a\varepsilon\rho}_{s\eta\sigma} - {}''\overset{*}{\Gamma}_\mu{}^\chi{}_\gamma T^{a\varepsilon\rho}_{b\chi\sigma} - {}''\Lambda_\sigma{}^\tau{}_\gamma T^{a\varepsilon\rho}_{b\eta\tau}) du^\gamma.$$

REFERENCES

|1| Dj.F.Nadj, *On subspaces of Riemann-Otsuki space. Publ. de l'Inst.Math. Beograd 30 (44) 1982* , 53-58

|2| T.Otsuki, *On general connection I.Math. Journal Okayama Univ. 9 (1959-60) 99-164.*

|3| P.K.Raschevski, *Riemannsche Geometrie und Tensoranalysis. Berlin 1959.*

REZIME

O ORTOGONALNIM PROSTORIMA PODPROSTORA RIEMANN-OTSUKIJEVOG PROSTORA

U paragrafu 1 dati su dovoljni uslovi da je potprostor S_m jednog Riemann-Otsuki-evog prostora isto $R\text{-}O_m$ i date su neke posledice ovog uslova. U paragrafu 2 uočeni su podprostori S_{n-m} ortogonalni na S_m, i odredjeni koeficijenti njihove koneksije i dokazano je da su uslovi iz paragrafa 1 dovoljni da bi S_{n-m} bio jedan Riemann-Otsuki-ev prostor.