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ON A STRUCTURE ϕ SATISFYING $(\phi^2+1)(\phi^2-a)=0$

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1. In |1| and |2| a unified notation is given of almost complex structure and almost contact structure by the introduction of a tensor field of type (1,1) on M^n such that $f^3+f=0$ and the rank $f=k$ is a constant everywhere The necessary and sufficient condition for an n-dimensional manifold to admit a tensor field $f\neq 0$ of type (1,1) such that $f^3+f=0$ is that k=2m and that the group of the tangent bundle of the manifold can be reduced to the group $U_{(m)}\times O_{(n-2m)}$. In |3| and |4| the structure $\phi^4\pm\phi^2=0$ is studied and the necessary and sufficient condition is given when M^n admits such a structure. In this paper, we want to introduce a tensor field of type (1,1) which satisfies the condition $(\phi^2+1)(\phi^2-a)=0$.

2. Let M^n be an n-dimensional differentiable manifold of the class C^∞ and let ϕ be a tensor field, $\phi^2\neq a$, $\phi^2\neq -1$ of type (1,1) and of class C^∞ such that n=2m,

$$(2.1)\qquad (\phi^2+1)(\phi^2-a)=0\ ,\quad a\in R^+\ ,\ a\neq 1$$

and rank $\phi=\frac{1}{2}$ (rank $\phi^2+\dim M^n$)=r=const.

For a differentiable manifold with a structure which satisfies such conditions we say that it asmits an $\phi(+1,a)$ structure.

Let

$$(2.2)\qquad \ell=\frac{\phi^2-a}{-1-a}\ ,\qquad m=\frac{\phi^2+1}{a+1}\ .$$

Then ℓ and m are complementary projection operators since $m^2 = m$, $\ell^2 = \ell$, $\ell m = m\ell = 0$, $\ell + m = 1$, which can easily be verified.

THEOREM 2.1. *Let ϕ satisfy conditions (2.1) and let ℓ and m be defined by condition (2.2). We then have*

$$\text{(2.3)} \qquad \phi\ell = \frac{\phi^3 - \phi a}{-1-a}, \qquad \phi m = \frac{\phi^3 + \phi}{a+1}$$

$$\phi^2\ell = \frac{-\phi^2 + a\phi^2 + a - a\phi^2}{-1-a} = \ell, \quad \phi^2 m = \frac{\phi^4 + \phi^2}{a+1} = \frac{\phi^2 + a\phi^2 + a - \phi^2}{a+1} = am.$$

Let L and M be complementary distributions which correspond to projections ℓ and m respectively. Then, (because of (2.3)), ϕ acts on L as an almost complex structure and on M as a structure for wich ϕ^2 is a homothety with coefficient of the homothety a. If ϕ is of constant rank r, then the dimensions of L and M are 2r-n and 2n-2r respectively. Obviously we have $n \leq 2r \leq 2n$.

REMARK 2.1. If the rank of ϕ is n, then $\phi^2 + 1 = 0$. Consequently, the $\phi(+1,a)$ structure of maximal rank is an almost complex structure.

REMARK 2.2. If the rank of ϕ is n/2, then $\phi^2 - a = 0$. Hence the $\phi(+1,a)$ structure of minimal rank is a structure for which ϕ^2 is a homothety with coefficient of the homothety a.

We shall now examine under which conditions the differentiable manifold admits a $\phi(+1,a)$ structure.

3. We now introduce a local coordinate system in the manifold and denote by ϕ_i^j, ℓ_i^j, m_i^j the local components of the tensors ϕ, ℓ, m respectively. We also introduce a positive definite Riemannian metric in the manifold and take 2r-n mutually orthogonal unit vectors v_a^i $(a,b,c,\ldots = 1,2,\ldots,2r-n)$ in L and 2(n-r) to be mutually orthogonal unit vectors v_A^j $(A,B,C,\ldots = 2r-n+1,\ldots,n)$ in M. We then have

$$(3.1)\qquad \begin{aligned} &\ell_i^j\, v_b^i = v_b^j\,, && \ell_i^j\, v_B^i = 0,\\ &m_i^j\, v_b^i = 0, && m_i^j\, v_B^i = v_B^j\,. \end{aligned}$$

If we denote by $(s_i^a,\ s_i^A)$ the matrix inverse to $(v_b^j,\ v_B^j)$ then s_i^a and s_i^A are both components of linearly indenpendent covariant vectors and satisfy the relations:

$$(3.2)\qquad s_i^a\, v_b^i = \delta_b^a\,,\quad s_i^a\, v_B^i = 0,\quad s_i^A\, v_b^i = 0\,,\quad s_i^A\, v_B^i = \delta_B^A\,,$$

$$(3.3)\qquad s_i^a\, v_a^j + s_i^A\, v_A^j = \delta_i^j\,.$$

If we put

$$(3.4)\qquad p_{ki} = s_k^a\, s_i^a + s_k^A\, s_i^A$$

then p_{ki} is a globally well-defined positive definite Riemannian metric with respect to which $(v_b^j,\ v_B^j)$ form an orthogonal frame such that

$$s_k^a = p_{ki}\, v_a^i,\quad s_k^A = p_{ki}\, v_A^i\,.$$

Now from (3.1) and (3.2) we find that

$$(\ell_i^j\, s_j^a)v_b^i = \delta_b^a\,,\qquad (\ell_i^j\, s_i^a)v_B^i = 0\,,$$

$$(m_i^j\, s_j^A)v_b^i = 0,\qquad (m_i^j\, s_j^A)v_B^i = \delta_B^A\,,$$

which show that:

$$(3.5)\qquad \begin{aligned} &\ell_i^j\, s_j^a = s_i^a\,, && m_i^j\, s_j^A = s_i^A\,,\\ &m_i^j\, s_j^a = 0\,, && \ell_i^j\, s_j^A = 0\,. \end{aligned}$$

On the other hand, from $\ell_i^j\, v_a^i = v_a^j$ we find that

$$\ell_k^j\, s_i^A\, v_a^k = s_i^a\, v_a^j\,,\qquad \ell_k^j(\delta_i^k - s_i^A\, v_A^k) = s_i^a\, v_a^j$$

that is,

$$(3.6)\qquad \ell_i^j = s_i^a\, v_a^j\,.$$

Similarly we get

$$(3.7) \qquad m_i^j = s_i^B \, v_B^i \, .$$

If we put

$$(3.8) \qquad \ell_{ki} = \ell_k^r \, p_{ri} \, , \quad m_{ki} = m_k^r \, p_{ri}^r \, ,$$

we find from (3.6), (3.7) and (3.4)

$$(3.9) \qquad \ell_{ki} = s_k^a \, s_i^a \, , \qquad m_{ki} = s_k^A \, s_i^A \, ,$$

$$(3.10) \qquad \ell_{ki} = \ell_{ik} \, , \qquad m_{ki} = m_{ik} \, , \quad \ell_{ik} + m_{ik} = p_{ik} \, .$$

We can also easily verify the following relations

$$(3.11) \qquad \ell_k^r \, \ell_i^q \, p_{rq} = \ell_{ki} \, , \quad \ell_k^r \, m_i^q \, p_{rq} = 0 \, ,$$
$$m_k^r \, m_i^q \, p_{rq} = m_{ki} \, .$$

For any two vectors x,y with components x^i, y^i let us put

$$(3.12) \qquad m^*(x,y) = m_{rq} x^r y^q \, , \; p(x,y) = p_{rq} x^r y^q \, ,$$

$$(3.13) \qquad \bar{g}(x,y) = \frac{1}{2}(p(x,y) + p(\phi x, \phi y) + m^*(x,y)) \, .$$

Then we have

$$m^*(v_A, v_a) = p(v_A, v_a) = 0,$$
$$\bar{g}(v_A, v_a) = \frac{1}{2}(p(v_A, v_a) + p(\phi v_A, \phi v_a) + m^*(v_A, v_a)) = 0 \, .$$

Thus L and M are orthogonal with respect to $\bar{g}$. Furthermore, it is easy to verify by using (3.8) and (3.10) that

$$p(\phi v_a, \phi v_b) = \ell_{rq} \phi_h^r \, \phi_j^q \, v_a^h \, v_b^j \, ,$$
$$p(\phi v_a, \phi v_b) + m^*(\phi v_a, \phi v_b) = p_{rq} \, \phi_h^r \, \phi_j^q \, v_a^h \, v_b^j \, ,$$
$$p(\phi^2 v_a, \, \phi^2 v_b) = p_{rq} \, v_a^r \, v_b^q \, .$$

These relations lead to the following:

$$\bar{g}(\phi x,\phi y) = \bar{g}(x,y) \quad \text{for all } x,y \text{ in } L. \tag{3.14}$$

Let M_1 be a space such that for $x \in M_1$, $\phi(x) = \sqrt{a}x$ and let M_2 be the distribution orthogonal to M_1 in M with respect to $\bar{g}$. We choose an orthonormal basis $u_{n-r+1},\dots,u_{2(n-r)}$ with respect to $\bar{g}$ for M_2. Furthermore, let $e_1,\dots,e_{2r-n}$ be an orthonormal basis for L with respect to $\bar{g}$. Using $\bar{g}$ we can defined a Riemannian metric g on M^n by

$$g(e_i,e_k) = \bar{g}(e_i,e_k),\quad g(e_i,u_\alpha) = \bar{g}(e_i,u_\alpha),\quad g(u_\alpha,u_\beta) = \bar{g}(u_\alpha,u_\beta)$$

$$g(e_i,\phi(u_\alpha)) = \bar{g}(e_i,\phi(u_\alpha)),\quad g(\phi(u_\alpha),u_\beta) = 0,\quad g(\phi(u_\alpha),\phi(u_\beta)) =$$

$$= \delta_{\alpha\beta},\quad 1 \le i,\ k \le 2r-n,\quad n-r+1 \le \alpha,\beta \le 2n-r\ .$$

Then g is well-defined because if $\bar{u}_{n-r+1},\dots,\bar{u}_{2(n-r)}$ is another orthonormal basis for M_2 then for $\bar{u}_\alpha = z_\alpha^\beta u_\beta$ we have

$$\delta_{\alpha\gamma} = \bar{g}(\bar{u}_\alpha,\bar{u}_\gamma) = \bar{g}(z_\alpha^\beta u_\beta,\ z_\gamma^\varepsilon u_\varepsilon) = z_\alpha^\beta z_\gamma^\varepsilon \delta_{\beta\varepsilon} = z_\alpha^\beta z_\gamma^\beta$$

and

$$g(\phi(\bar{u}_\alpha),\phi(\bar{u}_\gamma)) = g(z_\alpha^\beta \phi(u_\beta),z_\gamma^\varepsilon \phi(u_\varepsilon)) = z_\alpha^\beta z_\gamma^\varepsilon g(\phi(u_\beta),\phi(u_\varepsilon)) =$$
$$= z_\alpha^\beta z_\gamma^\beta = \delta_{\alpha\gamma}.$$

This means that there is a Riemannian metric g with respect to which L, M_1, M_2 are mutually orthogonal and

$$g(\phi x,\phi y) = g(x,y) \quad \text{for all } x,y \text{ in } L.$$
$$g(\phi x,\phi y) = a\cdot g(x,y) \quad \text{for all } x,y \text{ in } M.$$

THEOREM 3.1. *If in an* n*-dimension manifold* M^n $(n=2m)$ *a* $\phi(+1,a)$ *structure of rank* r *is given, then there exist complementary distributions* L *of dimension* $2r-n$, *and* M *of dimension* $2(n-r)$ *and a positive definite Riemannian metric* g *with respect to which* L *and* M *are orthogonal, and, furthermore, such that*

$$g(x,y) = g(\phi x,\phi y) \qquad x,y \in L$$
$$ag(x,y) = g(\phi x,\phi y) \qquad x,y \in M.$$

4. Take a vector e in the distribution L. Then the vector $\phi(e)$ is also in L and perpedikular to e, and moreover has the same lenght as e with respect to the metric g. Consequently we can choose 2r-n=2(r-m) orthonormal vectors in L such that

$$\phi(e_1) = e_{r-m+1}, \phi(e_2) = e_{r-m+2}, \ldots, \phi(e_{r-m}) = e_{2(r-m)},$$

and in M an orthonormal basis $e_{2(r-m)+1}, \ldots, e_n$ such that $e_{r+1}, \ldots, e_n$ are in M_2 and that $\phi(e_{r+1}) = -\sqrt{a}e_{2(r-m)+1}, \ldots, \phi(e_n) = -\sqrt{a}e_r$.

Then with respect to this orthonormal frame $\{e_1, \ldots, e_n\}$ the tensors g_{ji} and ϕ_j^i have components

$$(4.1) \quad g = \begin{bmatrix} E_{r-\frac{n}{2}} & 0 & 0 & 0 \\ 0 & E_{r-\frac{n}{2}} & 0 & 0 \\ 0 & 0 & E_{n-r} & 0 \\ 0 & 0 & 0 & E_{n-r} \end{bmatrix} \phi = \begin{bmatrix} 0 & E_{r-\frac{n}{2}} & 0 & 0 \\ -E_{r-\frac{n}{2}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{a}E_{n-r} & 0 \\ 0 & 0 & 0 & -\sqrt{a}E_{n-r} \end{bmatrix}$$

Such a frame is an adapted frame of the $\phi(+1,a)$ structure. Let $\{\bar{e}_i\}$ be another adapted frame in wich g and ϕ have the same components as (4.1).

Put $\bar{e}_i = \gamma_i^j e_j$. Then γ has a matrix of the form

$$\gamma = \begin{bmatrix} A_{r-\frac{n}{2}} & B_{r-\frac{n}{2}} & 0 & 0 \\ -B_{r-\frac{n}{2}} & A_{r-\frac{n}{2}} & 0 & 0 \\ 0 & 0 & C_{n-r} & 0 \\ 0 & 0 & 0 & D_{n-r} \end{bmatrix}$$

This means that the group of the tangent bundle of the manifold can be reduced to $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$. Conversely, if the group of the tangent bundle of the manifold can be reduced to $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$ then we can define a positive definite Riemannian metric g and a $\phi(+1,a)$ structure with matrices

(4.1) with respect to the adapted frames. Then we have

$$\phi^2 = \begin{bmatrix} -E_{r-\frac{n}{2}} & 0 & 0 & 0 \\ 0 & -E_{r-\frac{n}{2}} & 0 & 0 \\ 0 & 0 & aE_{n-1} & 0 \\ 0 & 0 & 0 & aE_{n-r} \end{bmatrix}, \phi^4 = \begin{bmatrix} E_{r-\frac{n}{2}} & 0 & 0 & 0 \\ 0 & E_{r-\frac{n}{2}} & 0 & 0 \\ 0 & 0 & a^2E_{n-r} & 0 \\ 0 & 0 & 0 & a^2E_{n-r} \end{bmatrix}$$

and it is easily verified that $(\phi^2+1)(\phi^2-a)=0$. From this we have

THEOREM 4.1. *A necessary and sufficient condition for the* n-*dimensional manifold to admit a* $\phi(+1,a)$ *structure is that the group of the tangent bundle can be reduces to the group* $U_{r-\frac{n}{2}} \times O_{n-r} \times O_{n-r}$.

It is known from |3| that if the structural group of a manifold M^n is reduced to $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$, then M^n admits a $\phi(4,+2)$ structure. L is the subspace of M^n on which $\phi^2=-1$, while the complement M of the space L in M^n admits an almost tangent structure.

From this paper it follows that if the structural group of manifold M^n is reduced to $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$, then M^n admits a $\phi(+1,a)$ structure. L is the subspace of M^n on which $\phi^2=-1$, while the complement M of space L in M^n admits the structure ϕ_M, for which $\phi_M^2=a$.

From this we have:

THEOREM 4.2. *If the structural group of manifold* M^n *is reduced to* $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$, *then* M^n *admits a* $\phi(4,+2)$ *and* $\phi(+1,a)$ *structure.* L *is the subspace of* M^n *on which* $\phi^2=-1$ *for both structures. Complement* M *of the space* L *in* M^n *admits structure for which* $\phi_M^2=0$ *or* $\phi_M^2=a$.

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REZIME

O STRUKTURI ϕ KOJA ISPUNJAVA USLOV $(\phi^2+1)(\phi^2-a)=0$

Potreban i dovoljan uslov da se n-dimenzionalna mnogostrukost može snabdeti $\phi(+1,a)$ strukturom je da se grupa tangentnog bandla može reducirati do $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$. Ako se strukturna grupa može reducirati do $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$ tada se mnogostrukost može snabdeti i $\phi(4,+2)$ i $\phi(+1,a)$ strukturom. U prvom slučaju je $\phi_L^2=-1$, a na komplementarnom prostoru M je $\phi_M^2=0$. U drugom slučaju $\phi_L^2=-1$, a na M je $\phi_M^2=a$.