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TRANSITIVE n-ARY RELATIONS AND CHARACTERIZATIONS OF GENERALIZED EQUIVALENCES*)

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Pickett |2| defines generalized equivalence relations and relates them to the partitions of type n, given by Hartmanis |1|. In this article several types of generalized reflexive, symmetric and also transitive relations are defined and properties and connections between some of these relations are given. Finally, some characterization theorems for generalized equivalence relations are proved.

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1. (n+1)-ary relation R on the set $S\neq\emptyset$ is (i,j)-reflexive, $i\neq j$, $i,j\in\{1,\dots,n+1\}$, iff

$$(\forall a_1,\dots,a_{j-1},a_{j+1},\dots,a_{n+1}\in S)\ ((a_1^{i-1},\ a_i,a_{i+1}^{j-1},a_i,a_{j+1}^{n+1})\in R).^{1)}$$

R is reflexive iff it is (i,j)-reflexive for all $i,j\in\{1,\dots,n+1\}$, $i\neq j$.[2)]

2. (n+1)-ary relation R on S is π-symmetric, $\pi\in\{1,\dots,n+1\}!$,[3)] iff

$$(\forall a_1,\dots,\ a_{n+1}\in S)((a_1^{n+1})\in R\Rightarrow\ (a_{\pi(1)},\dots,a_{\pi(n+1)})\in R).$$

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*) Presented april, 27, 1981. 1) a_p^q stands for $a_p,a_{p+1},\dots,a_{q-1},a_q$, and denotes an empty syllable when $q<p$; consequently a_p^p is a_p, and instead of a, a,...,a (n times), we write $\overset{n}{a}$; $\overset{0}{a}$ is, clearly, empty.

2) In |2| (n+1)-ary (1,n+1)-reflexive relation is called "reflexive"; in |3| the term "strongly reflexive" is used for the reflexive relations.

3) If M finite, M! is a set of all permutations on M.

(|2|) R is symmetric iff it is π-symmetric for all $\pi \in \{1,\dots,n+1\}!$.

3. Let R be (n+1)-ary relation on S and ${}^1a_1,\dots,{}^1a_{n+1},\dots,{}^ka_1,\dots,{}^ka_{n+1}, b_1,\dots,b_{n+1}$ variables. Let also

1) $k \in N \setminus \{1\}$;

2) α is the k-ary relation on the set R; and

3) (b_1^{n+1}) is taken by the given nullary operation in $\{{}^1a_1,\dots,{}^1a_{n+1},\dots,{}^ka_1,\dots,{}^ka_{n+1}\}^{n+1}$.

R now belongs to the class of transitive relations iff the following implication is satisfied:

$$({}^1a_1^{n+1}) \in R \wedge \dots \wedge ({}^ka_1^{n+1}) \in R \wedge \wedge (({}^1a_1^{n+1}),\dots,({}^ka_1^{n+1})) \in \alpha \Rightarrow (b_1^{n+1}) \in R, \tag{1}$$

for all ${}^1a_1,\dots,{}^1a_{n+1},\dots,{}^ka_1,\dots,{}^ka_{n+1} \in S$.[1)]

In this article we shall be concerned with some relations belonging to this class, with k=2, and k=n+1.

3_1) (n+1)-ary relation R on S is iA_1-transitive, $i \in \{1,\dots,n\}$. iff

$$(\forall a_o,\dots,a_{n+1} \in S)((a_o^{i-1},a_i,a_{i+1}^n) \in R \wedge (a_1^{i-1},a_i,a_{i+1}^{n+1}) \in R \Rightarrow \Rightarrow (a_o^{i-1}, a_{i+1}^{n+1}) \in R). \tag{2}$$

(n+1)-ary relation R on S is iA_1^*-transitive, $i \in \{1,\dots,n\}$, iff

$$(\forall a_o,\dots,a_{n+1} \in S)((a_o^{i-1},a_i a_{i+1}^n) \in R \wedge (a_1^{i-1},a_i a_{i+1}^{n+1}) \in R \wedge \wedge (a_j \neq a_i, \text{ for } j \in \{1,\dots,n\} \setminus \{i\}) \Rightarrow (a_o^{i-1},a_{i+1}^{n+1}) \in R). \tag{3}$$

(n+1)-ary relation R on S is $i\bar{A}_1$-transitive,[*)] $i \in \{1,\dots,n\}$, iff

$$(\forall a_o,\dots,a_{n+1} \in S)((a_o^{i-1},a_i a_{i+1}^n) \in R \wedge (a_1^{i-1},a_i a_{i+1}^{n+1}) \in R \wedge$$

1) In the transitivities considered here, α shall always be such that for n=1 (1) reduces to the usual transitivity low.

*) In |2|: "transitive" stands for "$n\bar{A}_1$-transitive.

(4) $(a_j \neq a_k,$ for $j=k,\ j,k \in \{1,\dots,n\}) \Rightarrow (a_o^{i-1}, a_{i+1}^{n+1}) \in R)$.

$3_2)$ (n+1)-ary relation R on S is $i\bar{A}_2$-transitive, [2] $i \in \{1,\dots,n\}$ iff

$$(\forall a_o,\dots,a_{n+1} \in S)((a_o, a_1^{i-1}, a_i, a_{i+1}^n) \in R \wedge (a_i, a_1^{i-1}, a_{i+1}^{n+1}) \in R \wedge$$

$$(a_j \neq a_k, \text{ for } j \neq k, j,k \in \{1,\dots,n\}) \Rightarrow (a_o, a_1^{i-1}, a_{i+1}^{n+1}) \in R).$$

$3_3)$ (n+1)-ary relation R on S is $i\bar{M}_1$-transitive, $i \in \{2,\dots,n+1\}$, iff

$$(\forall a_1,\dots,a_{i-1},a_{i+1},\dots,a_{n+1},x_i^{(1)},\dots,x_i^{(n)},y \in S)((x_i^{(j)} \neq$$

$$\neq x_i^{(k)}, \text{ for } j\neq k,\ j,k \in \{1,\dots,n\}) \wedge \bigwedge_{s=1}^{n} (a_1^{i-1}, x_i^{(s)}, a_{i+1}^{n+1}) \in R \wedge$$

$$(x_i^{(1)},\dots,x_i^{(n)},y) \in R \Rightarrow (a_1^{i-1}, a_{i+1}^{n+1}, y) \in R .$$ [3]

$3_4)$ (n+1)-ary relation R on S is $i\bar{M}_2$-transitive, $i \in \{2,\dots,n+1\}$ iff

$$(\forall a_1,\dots,a_{i-1},a_{i+1},\dots,a_{n+1},x_i^{(1)},\dots,x_i^{(n)},y \in S)((x_i^{(j)} \neq$$

$$\neq x_i^{(k)}, \text{ for } j\neq k,\ j,k \in \{1,\dots,n\}) \wedge \bigwedge_{s=1}^{n} (a_1^{i-1}, x_i^{(s)}, a_{i+1}^{n+1}) \in R \wedge$$

$$(x_i^{(1)},\dots,x_i^{(n)},y) \in R \Rightarrow (a_1^{i-1}, y, a_{i+1}^{n+1}) \in R).$$

REMARK.

In the case n=1 all notions defined in 1), 2) and 3) reduce to the usual binary notions.

$4_1)$ |1| For set S with at least n elements, the family P_n of subsets of S is a partition of type n, iff (1) each member of P_n has at least n elements and (2) each n different elements

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2) The notion of an $i\bar{A}_2$-transitive relation is from|4|. It is obvious that one can define transitive relations without or with one star in $3_2)$, $3_3)$ and $3_4)$ as in $3_1)$. It has not been done here since the purpose of this atricle is to treat the transitivities connected with generalized equivalence relations.

3) M-transitivities appeared in investigation of generalized orderings. $\bigwedge_{i=1}^{n} P_i$ denotes a logical conjunction $P_1 \wedge \dots \wedge P_n$.

S belong to exactly one member of P_n.

4_2) |2| (n+1)-ary relation E_n on S is a generalized equivalence relation on S iff it satisfies:

E1n: (1,n+1)-reflexivity,

E2n: symmetry, and

E3n: $n\bar{A}_1$-transitivity.

4_3) In |2| it is shown that (n+1)-ary (i.e. generalized) equivalence relation E_n on S induces on S a partition of type n, and contrary, that each partition of type n on S can be connected with the generalized, (n+1)-ary equivalence relation on the same set.

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PROPOSITION 1. *If* (n+1)*-ary relation* R *on* S *is* (1, i+1)*-reflexive and* $i\bar{A}_1$ *-transitive, then it is* $i\bar{A}_2$ *-transitive.*

P r o o f. Let

a) $(x_o,x_1^{i-1},x_i,x_{i+1}^n) \in R$ and

b) $(x_i,x_1^{i-1},x_{i+1}^n,x_{n+1}) \in R$,

where $x_i \neq x_j$ for $i \neq j$, $i,j \in \{1,\dots,n\}$. Then

c_1) $(x_{i-1},x_i,x_1^{i-2},x_{i-1},x_{i+1}^n) \in R$ ((1,i+1)-reflexivity).

From (c_1) and (b), by $i\bar{A}_1$-transitivity, it follows

b_1) $(x_{i-1},x_i,x_1^{i-2},x_{i+1}^n,x_{n+1}) \in R$. Applying $i\bar{A}_1$-transitivity on (b_1) and

c_2) $(x_{i-2},x_{i-1},x_i,x_1^{i-3},x_{i-2},x_{i+1}^n) \in R$ ((1,n+1)-reflexivity),

we get

b_2) $(x_{i-2},x_{i-1},x_i,x_1^{i-3},x_{i+1}^{n+1}) \in R$.

This procedure leads to the conclusion

($\bar{b}$) $(x_1^{i-1},x_i,x_{i+1}^{n+1}) \in R$. Finally, from (a) and ($\bar{b}$), by $i\bar{A}_1$-transitivity it follows that

$(x_o, x_1^{i-1}, x_{i+1}^{n+1}) \in R$, which was to be proved.

COROLLARY 2. *(n+1)-ary (1,n+1)-reflexive and* $n\bar{A}_1$*- transitive relation* R *on* S *is* $n\bar{A}_2$ *-transitive.*

REMARK 2.

(1,n+1)-reflexivity and $n\bar{A}_2$-transitivity do not imply $n\bar{A}_1$-transitivity, which can be shown by the following example, for n=2. R is a ternary relation on {a,b,c,d} consisting of all triples with equal first and third coordinates and of (a,b,c), (c,b,d), (a,b,d) and (b,d,a). It is easy to check that R satisfies (1,3)-reflexivity and $2\bar{A}_2$-transitivity, but that it is not $2\bar{A}_1$-transitive.

The following corollary is a consequence of the proof of Proposition 1.

COROLLARY 3. *(n+1)-ary (1,i+1) -reflexive and* $i\bar{A}_1$ *- transitive relation* R *on* S *satisfies the property*

$$(\gamma_i) : (\forall a_1,\dots,a_{n+1} \in S)((a_j \neq a_k, j\neq k,\ j,k \in \{1,\dots,n\}) \wedge$$
$$(a_1^{n+1}) \in R \Rightarrow (a_{\gamma_i(1)},\dots,a_{\gamma_i(n+1)}) \in R),$$
$$\gamma_i = (i,1,\dots,i-1,i+1,\dots,n+1) \in \{1,\dots,n+1\}!.$$

PROPOSITION 4. *If (n+1)-ary (1,i+1)-reflexive and* $i\bar{A}_2$ *transitive relation* R *on* S *satisfies* (γ_i), *then* R *is* $i\bar{A}_1$ *-transitive* $(i \in \{1,\dots,n\})$.

P r o o f. If

$$(x_o^{i-1}, x_i, x_{i+1}^n) \in R \text{ and } (x_1^{i-1}, x_i, x_{i+1}^{n+1}) \in R\ ,\ x_i \neq x_j \text{ for } i\neq j,$$

$i,j \in \{1,\dots,n\}$, then by (γ_i) it follows that

$$(x_o^{i-1}, x_i, x_{i+1}^n) \in R \quad\text{and}\quad (x_i x_1^{i-1}, x_{i+1}^{n+1}) \in R\ .$$

Thereby $i\bar{A}_2$-transitivity implies

$(x_o^{i-1}, x_{i+1}^{n+1}) \in R$, completing the proof of the lemma.

Proposition 1, Corollary 3 and Proposition 4 imply the following proposition.

PROPOSITION 5. *If* (n+1)-*ary* (1,n+1) -*reflexive relation* R *on* S *satisfies* (γ_i), *then* R *is* $i\bar{A}_2$ -*transitive iff it is* $i\bar{A}_1$ -*transitive*, $(i\in\{1,\dots,n\})$.

PROPOSITION 6. *If* (n+1)-*ary relation* R *on* S *satisfies* $(i+1)\bar{M}_1$-*transitivity* (i=1,...,n), (j,i+1)-*reflexivity for all* $j\in\{2,\dots,i\}$ *and* (i+1,k) -*reflexivity for all* $k\in\{i+2,\dots,n+1\}$, *then* R *satisfies* $i\bar{A}_1$-*transitivity.*

P r o o f. Let $(x_o^n)\in R$ and $(x_1^{n+1})\in R$, $x_u\neq x_v$, $u\neq v$, $u,v\in\{1,\dots,n\}$. Then

$(x_o,x_1,\dots,x_{i-1},x_1,x_{i+1},\dots,x_n)\in R$ ((2,i+1)-reflexivity),

$(x_o,x_1,\dots,x_{i-1},x_2,x_{i+1},\dots,x_n)\in R$ ((3,i+1)-reflexivity),

. . . .

$(x_o,x_1,\dots,x_{i-1},x_{i-1},x_{i+1},\dots,x_n)\in R$ ((i,i+1)-reflexivity),

$(x_o,x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n)\in R$ (by assumption),

$(x_o,x_1,\dots,x_{i-1},x_{i+1},x_{i+1},\dots,x_n)\in R$ ((i+1,i+2)-reflexivity),

. . . .

$(x_o,x_1,\dots,x_{i-1},x_n,x_{i+1},\dots,x_n)\in R$ ((i+1,n+1)-reflexivity),

$(x_1,\dots,x_{i-1},x_n,x_{i+1},\dots,x_n,x_{n+1})\in R$ (by assumption).

Thereby $(i+1)\bar{M}_1$-transitivity implies

$(x_o,x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,x_{n+1})\in R$, which was to be proved.

COROLLARY 7. *If* (n+1)-*ary* $(n+1)\bar{M}_2$-*transitive relation* R *on* S *satisfies*(i,n+1) -*reflexivity for all* $i\in\{2,\dots,n\}$, *then* R *satisfies* $n\bar{A}_1$-*transitivity.*

REMARK 3.

The following example is a reflexive, $n\bar{A}_1$-transitive ternary relation on {a,b,c,d,e}, which is not $(n+1)\bar{M}_2$-transitive. Let R consist of all triples with at least two equal co - ordinates and of (a,b,c),(b,a,c),(a,b,d),(b,a,d),(c,d,e) and (d,c,e). It is obvious that R is reflexive, $2\bar{M}_2$-transitive, but not $3\bar{M}_2$-transitive, since $(a,b,c) \in R$, $(a,b,d) \in R$, $(c,d,e) \in R$, but $(a,b,e) \notin R$.

COROLLARY 8. *If* (n+1)*-ary relation* R *on* S *satisfies* $(i+1)\bar{M}_2$*-transitivity* (i=1,...,n), (j,i+1)*-reflexivity for all* $j \in \{2,\dots,n\}$, (i+1,k) *-reflexivity for all* $k \in \{i+2,\dots,n+1\}$ *and* γ*-symmetry for* $\gamma = (1,\dots,i,n+1,i+1,\dots,n) \in \{1,\dots,n+1\}!$, *then* R *satisfies* $i\bar{A}_1$*-transitivity.*

PROPOSITION 9. *If* (n+1)*-ary relation* R *on* S *satisfies* (1,j)*-reflexivity for all* $j \in \{2,\dots,i+1\}$, $i\bar{A}_1$*- and* $(i-1)\bar{A}_1$*-transitivity,* $i \in \{1,\dots,n\}$, *then* R *is* $(i-2)\bar{A}_1$*-transitive* ($i-2,i-1,i \in \{1,\dots,n\}$) .

P r o o f.

(a) $(a_0^{i-3},a_{i-2},a_{i-1}^n) \in R$ and $(a_1^{i-3},a_{i-2},a_{i+1}^{n+1}) \in R$, $a_i \neq a_j$, $i \neq j$, $i,j \in \{1,\dots,n\}$.

i) Suppose first $a_0 \neq a_i$, i=1,...,n. Then, by using the well known properties of permutations, by Corollary 3 and since R is $i\bar{A}_1$-, $(i-1)\bar{A}_1$-transitive and (1,i)-reflexive, (a) implies

($\bar{a}$) $(a_0^{i-1},a_{i-2},a_i^n) \in R$ and $(a_1^{i-1},a_{i-2},a_i^{n+1}) \in R$.

By $(i-1)\bar{A}_1$-transitivity then $(a_0^{i-3},a_{i-1}^{n+1}) \in R$.

(ii) Let $a_0 \neq a_{i-2}$, and

($\bar{b}$) $(a_{i-2},a_1^{i-3},a_{i-2},a_{i-1}^n) \in R$ and $(a_1^{i-3},a_{i-2},a_{i-1}^{n+1}) \in R$.

By using (1,i)-reflexivity and by the procedure used in i),

($\bar{c}$) $(a_{i-2},a_1^{i-3},a_{i-1},a_{i-2},a_i^n) \in R$ and

($\bar{d}$) $(a_1^{i-3},a_{i-1},a_{i-2},a_i^{n+1}) \in R$.

From ($\bar{c}$) and ($\bar{d}$), by $(i-1)\bar{A}_1$-transitivity, if follows that

$(a_{i-2},a_1^{i-3},a_{i-1},a_1^{n+1}) \in R$.

iii) Let $a_o = a_j$, $j \in \{1,\dots,i-3,i-1,\dots,n\}$, and

$$(a_t, a_1^{i-3}, a_{i-2}, a_{i-1}^{t-1}, a_t, a_{t+1}^{n}) \in R \quad \text{and} \quad (a_1^{i-3}, a_{i-2}, a_{i-1}^{t-1}, a_t, a_{t+1}^{n+1}) \in R,$$

where $t < i$ or $t > i$, $t \in \{1,\dots,n\}$. (1,t+1)-reflexivity now gives

$$(a_t, a_1^{i-3}, a_{i-1}, a_i^{t-1}, a_t, a_{t+1}^{n+1}) \in R .$$

i), ii) and iii) prove the proposition.

Using the fact that each permutation on $\{1,\dots,n+1\}$ can be produced by two cycles $\gamma_{n+1} = (n+1,1,\dots,n)$ and $\gamma_n = (n+1,\dots$ $\dots,n-1,n+1)$, one can easely show that the following propositi-on is a consequence of the previously proved statements.

PROPOSITION 10. *If (n+1)-ary relation R on S satisfies γ_{n+1} and γ_n-symmetry (γ_{n+1} and γ_n are given above), then*

1) R *is reflexive iff it is* (1,n+1)-*reflexive ;*

2) R *is* $i\bar{A}_1$ *-transitive,* $i \in \{1,\dots,n\}$, *iff it is* $n\bar{A}_1$*- transitive ;*

3) R *is* $i\bar{A}_2$ *-transitive,* $i \in \{1,\dots,n\}$, *iff it is* $n\bar{A}_2$*-transitive ;*

4) R *is* $i\bar{M}_1$ *-transitive,* $i \in \{2,\dots,n+1\}$, *iff it is* $(n+1)\bar{M}_1$ *-transitive ;*

5) R *is* $i\bar{M}_2$*-transitive,* $i \in \{2,\dots,n+1\}$, *iff it is* $(n+1)\bar{M}_2$*-transitive ;*

6) R *is* $n\bar{A}_2$*-transitive, iff it is* $n\bar{A}_1$*-transitive.*

PROPOSITION 11. *(n+1)-ary relation R on S is the generalized equivalence relation on S in the sense of* 4_2*) iff*

I R *is reflexive ;*

II R *satisfies the property*

$$(\tau) \quad (\forall a_1,\dots,a_{n+1} \in S)((a_1^{n+1}) \in R \wedge (a_i \neq a_j,\ i \neq j,\ i,j \in \{1,\dots,n\})$$

$$\Rightarrow (a_{\tau(1)},\dots,a_{\tau(n+1)}) \in R, \ \textit{where}\ \tau = (n+1,2,\dots,n,1) ;$$

III R *is* $n\bar{A}_1$*-transitive.*

P r o o f. The only nontrivial part of the proof is the one in which the symmetry has to be proved, under the assumption that R satisfies I, II and III.

Suppose $(a_1^{n+1}) \in R$.

If $a_i = a_j$, for some $i \neq j$, $i,j \in \{1,\dots,n+1\}$, then this part of the symmetry follows directly from I and the definition of reflexivity.

If there are no equal elements among a_i, $i \in \{1,\dots,n+1\}$, then Corollary 3, I, and III imply that if

$(a_1^{n+1}) \in R$ then all (n+1)-tiples produced by the cycle $(n,1,\dots,n-1,n+1)$ also belong to R. Since this cycle and τ produce each permutation on $\{1,\dots,n+1\}$, using II, we get that for each permutation $\pi \in \{1,\dots,n+1\}$!

$(a_{\pi(1)},\dots,a_{\pi(n+1)}) \in R$, proving that T satisfies the symmetry.

There is another characterization of the generalized equivalence relation, depending on reflexivity and two transitivities.

PROPOSITION 12. *(n+1)-ary relation R on S is a generalized equivalence relation in the sense of 4_2) iff*

(α) R *is reflexive ;*

(β) R *is* $(n-1)\bar{A}_1$ *-transitive ;*

(γ) R *is* $n\bar{A}_1$ *-transitive .*

P r o o f. Let $(a_o, a_1^{n-2}, a_{n-1}, a_n) \in R$. By $(n-1,n+1)$-reflexivity $(a_1^{n-2}, a_{n-1}, a_n, a_{n-1})$ also belongs to R. Thereby, for different $a_1,\dots,a_{n+1}$, (β) implies that if $(a_1^{n+1}) \in R$, then $(a_1^{n-1}, a_{n+1}, a_n) \in R$. The proof of the symmetry now follows from the procedure used in proving Proposition 11.

The following proposition shows that, assuming $(1,n+1)$-reflexivity and some of the symmetry, $i\bar{A}_1$-transitivity implies $(i+1)\bar{M}_1$-transitivity. Putting together this statement and Proposition 6., one gets the conditions for the equivalence of these two transitivies.

PROPOSITION 13. *If (n+1)-ary relation* R *on* S *satisfies* γ_{n+1} *- and* γ_n *-symmetry* ($\gamma_{n+1}=(n+1,1,\dots,n)$, $\gamma_n=(n,1,\dots,n-1,n+1)$), $i\bar{A}_1$ *-transitivity* ($i\in\{1,\dots,n\}$) *and* $(1,n+1)$ *-reflexivity, then* R *satisfies* $(i+1)\bar{M}_1$ *-transitivity.*

P r o o f. 1) R is symmetric, because it satisfies γ_{n+1}- and γ_n- symmetry (see the text preceding Proposition 10.).

2) Let $(a_1^n,x_1)\in R,\dots,(a_1^n,x_n)\in R$, $x_i\neq x_j$ for $i\neq j$, $i,j\in\{1,\dots,n\}$. Now, if there are equal elements among $a_1,\dots,a_n,y$, reflexivity implies $(a_1^n,y)\in R$, proving $(i+1)\bar{M}_1$-transitivity for R.

Assume now that $a_i\neq a_j$. $a_i\neq y$ $i,j\in\{1,\dots,n\}$. Also let $a_i=x_i$, $i=1,\dots,k$, $k\in\{0,\dots,n\}$ (for k=0 there are no such equal elements). Assumption 2) is now given by

(a) $(x_1^k,a_{k+1}^n,x_1)\in R,\dots,(x_1^n,a_{k+1}^n,x_k)\in R$,

(b) $(x_1^k,a_{k+1}^n,x_{k+1})\in R,\dots,(x_1^k,a_{k+1}^n,x_n)\in R$.

From (b), by symmetry, it follows that

$$(x_{k+1},x_1^k,a_{k+1}^n)\in R \text{ and } (x_1^k,a_{k+1}^n,x_t)\in R,\quad t\in\{k+2,\dots,n\}.$$

$n\bar{A}_1$ -transitivity now implies

(c) $(x_{k+1},x_1^k,a_{k+1}^{n-1},x_t)\in R$, $t\in\{k+2,\dots,n\}$.

From this, by symmetry and $n\bar{A}_1$-transitivity, it follows that

(d) $(x_{k+2},x_{k+1},x_1^k,a_{k+1}^{n-2},x_v)\in R$, $v\in\{k+3,\dots,n\}$.

Continuing this procedure, we finally get $(x_1^n,a_{k+1})\in R$, i.e.

(e) $(a_{k+1},x_1^n)\in R$.

Since $x_i=a_i$, $i=1,\dots,k$, from (a), (b) and (e) it follows that $(a_{k+1},a_1^k,x_{k+1}^n)\in R$ and $(a_1^k,x_{k+1}^n,y)\in R$, and applying $n\bar{A}_1$-transitivity, we get

(f) $(a_{k+1},a_1^k,x_{k+1}^{n-1},y)\in R$.

Applying $n\bar{A}_1$-transivity on

$(a_{k+2}, a_{k+1}, a_1^k, x_1^{n-1}) \in R$, $(a_{k+1}, a_1^k, x_1^{n-1}, y) \in R$,

(the first (n+1)-tiple is from (d) i.e. from the procedure exposed there, and the second is (f)), we get

(g) $(a_{k+2}, a_{k+1}, a_1^k, x_{k+1}^{n-2}, y) \in R$.

Continuing this procedure, we finally get

$(a_n, a_{n-1}, \ldots, a_{k+1}, a_1^k, y) \in R$, i.e. $(a_1^n, y) \in R$, proving the statement, if $i=n$.

In the case when $i \neq n$, suppose that

(h) $(a_1^{n-1}, x_1, a_{i+1}^{n+1}) \in R, \ldots, (a_1^{i-1}, x_n, a_{i+1}^{n+1}) \in R$ and

(i) $(x_1^n, y) \in R$, $x_i \neq x_j$, for $i \neq j$, $i,j \in \{1,\ldots,n\}$.

Since R is symmetric (1)), we have (from (h) and (i))

(j) $(a_1^{i-1}, a_{i+1}^{n+1}, x_1) \in R, \ldots, (a_1^{i-1}, a_{i+1}^{n+1}, x_n) \in R$ and

(k) $(x_1^n, y) \in R$, $x_i \neq x_j$, for $i \neq j$, $i,j \in \{1,\ldots,n\}$.

Applying the symmetry and 4), Proposition 10, the proof in this case is the same as the one given for $i=n$.

COROLLARY 14. *If* (n+1)*-ary reflexive relation on* S *satisfies* $(n-1)\bar{A}_1$*- and* $n\bar{A}_1$*-transitivity, then it is* $(n+1)\bar{M}_1$ *-transitive.*

P r o o f. This is a consequence of the two previous propositions.

COROLLARY 15. (n+1) *-ary* (1,n+1)*-reflexive and symmetric relation* R *on* S *is* $n\bar{A}_1$*-transitive iff it is* $(n+1)\bar{M}_1$*-transitive.*

P r o o f. Immediately by Proposition 6. and Proposition 13.

REMARK 4.

Applying Corollary 8., and the first part of the proof of Proposition 13. concerning the symmetry, one can put "(i+1) $\bar{M}_2$-" instead of "(i+1)$\bar{M}_1$-" into the formulation of Proposition 13., which will remain true.

* * *

In this part we shall show that the condition of n different elements in i$\bar{A}_1$-transitivity ((4) in (3_1), given by Pickett |2| can be weakened ((3) in 3_1) , giving something new in the previous characterizations.

The following lemma follows immediately from the definitions of iA_1^*- and i$\bar{A}_1$-transitivity.

LEMMA 16. *a)* iA_1^* *-transitive relation is* i$\bar{A}_1$*-transitive;*

b) For n=2 *the relation* R *is* i$\bar{A}_1$*-transitive iff it is* i$\bar{A}_1^*$*-transitive (i.e. these two definitions to not differ).*

REMARK 5.

If we consider reflexive relations, the condition $a_j \neq a_i$ could not be weakened more, since for example, for n=2, from (a,b,b) ∈ R and (b,b,c) ∈ R, its absence implies (a,b,c) ∈ R. The reflexive relations would thus always consist of all triples ((n+1)- tiples) on the given set.

PROPOSITION 17. (n+1)*-ary relation* R *on* S $\|S\| \geq n$)*,is the generalized equivalence relation on* S *in the sense of* 4_2) *iff it satisfies:*

(i) for each sequence of n *different elements* $x, y_1, \ldots$ $\ldots, y_{n-1}$ *of* S*, there is* y *in* S, $y = y_{n-1}$*, such that*

$$(x, y_1^{n-1}, y) \in R \;;$$

(ii) R *is* γ_{n+1}*-and* γ_n*-symmetric, where*

$\gamma_{n+1} = (n+1,1,\ldots,n)$ *and* $\gamma_n = (n,1,\ldots,n-1,n+1)$[1) ;

(iii) R *is* nA_1^* - *transitive.*

REMARK 6.

For n=1, (i) reduce to the statement that for each $x \in S$, there is $y \in S$, such that $(x,y) \in R$, and this is equivalent to the usual condition $\mathcal{D}R = S$, in the binary case.

Proof of Proposition 17: We have to prove that R satisfies (1,n+1)-reflexivity (the rest is trivial), if it satisfies (i), (ii) and (iii).

Let $x_o,\ldots,x_{n-1} \in S$, $x_i \neq x_j$, $i,j \in \{0,\ldots,n-1\}$. Then by (i), there is $y \in S$, $y \neq x_{n-1}$, and $(x_o^{n-1},y) \in R$. From here, by symmetry, it follows that $(x_1^{n-1},y,x_o) \in R$, and by (iii)

$$(x_o^{n-1},x_o) \in R, \text{i.e. (symmetry)}, \quad (\overset{2}{x}_o,x_1^{n-1}) \in R.$$

From $(x_o^{n-1},y) \in R$ and $(\overset{2}{x}_o,x_1^{n-1}) \in R$, (iii) implies

$$(\overset{2}{x}_o,x_1^{n-2},x_o) \in R \text{ i.e. } (\overset{3}{x}_o,x_1^{n-2}) \in R.$$

Continuing this procedure, we finally get

(a) $(\overset{i}{x}_o,x_1^{n-(i-1)}) \in R$, for each $i \in \{2,\ldots,n+1\}$, and for arbitrary different $x_o,\ldots,x_{n-1} \in S$.

Applying the symmetry on (a), we get

$(x_1,\overset{i}{x}_o,x_2^{n+1-i}) \in R$, and $(\overset{i}{x}_o,x_2^{n+1-i},x_1) \in R$, and (iii) gives $(x_1,\overset{i}{x}_o,x_2^{n-i},x_1) \in R$ i.e. $(\overset{2}{x}_1,\overset{i}{x}_o,x_2^{n-i}) \in R$, and again, by (iii), $(\overset{3}{x}_1,\overset{i}{x}_o,x_2^{n-i-1}) \in R$.

Continuing, we get

$$(\overset{i}{x}_o,\overset{j}{x}_1,x_2^{n+2-i-j}) \in R, \quad i+j \leq n+1.$$

The same procedure gives

1) γ_{n+1}-and γ_n-symmetry produce the (whole) symmetry. Instead of γ_n one can take an arbitrary transposition.

$$(x_o^{i_o}, x_1^{i_1}, \dots, x_j^{i_j}, x_{j+1}^{n+j+1-i_o-\dots-i_j}) \in R \ , \ i_o+\dots+i_j \leq n+1 \ ,$$

completing by (ii), the proof of the reflexivity.

REMARK 7.

Note that the "only if" part of the proof of the preceding proposition shows that "some" of nA_1^*-transivity is included in reflexivity, in the case when there are equal element among $x_1, \dots, x_{n-1}$, for $(x_o^n) \in R$ and $(x_1^{n+1}) \in R$.

The fact that reflexivity does much more in generalized equivalences than in the binary case, can be shown by the following example, for n=2. Let R consists of all triples with equal coordinates (i.e. for $x \in S$, $(x,x,x) \in R$), and of arbitrary triples with different coordinates ($(x_1,x_2,x_3) \in R$, $x_i \neq x_j$, $i \neq j$), provided that R is $2\bar{A}_1$-transitive. Then no part of the symmetry can be produced, since there are no triples in R, being of the form (x,y,x), $x \neq y$.

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REZIME

TRANZITIVNE n-ARNE RELACIJE I KARAKTERIZACIJE UOPŠTENIH EKVIVALENCIJA

Pickett |2| definiše uopštene relacije ekvivalencije i povezuje ih sa particijama tipa n koje je uveo Hartmanis |1|. U ovom radu dati su različiti tipovi uopštenih refleksivnih , simetričnih, kao i tranzitivnih relacija. Ispitane su osobine tih relacija i data su tvrdjenja koja ih povezuju. Najzad, dokazani su stavovi o različitim karakterizacijama uopštenih ekvivalencija.