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# A FIXED POINT THEOREM IN PROBABILISTIC METRIC SPACES

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ABSTRACT

In this paper a fixed point theorem in probabilistic metric spaces is proved. This theorem is a generalization of a fixed point theorem from |3|.

A pair $(S,F)$ is a probabilistic metric space if and only if $S$ is an arbitrary set, $F: SXS \to \Delta^+$ ($\Delta^+$ is the set of all the distribution functions $F$ such that $F(0)=0$) so that the following conditions are satisfied ($F(x,y)=F_{x,y}$ for every $x,y \in S$):

1. $F_{x,y}(\varepsilon) = 1$, for every $\varepsilon \in R^+$ iff $x=y$.
2. $F_{x,y} = F_{y,x}$, for every $x,y \in S$.
3. $F_{x,y}(\varepsilon) = 1$ and $F_{y,z}(\delta) = 1$ implies $F_{x,z}(\varepsilon+\delta) = 1$,

where $x,y,z \in S$, $\varepsilon,\delta \in R^+$.

The $(\varepsilon,\lambda)$-topology in $S$ is introduced by the $(\varepsilon,\lambda)$-neighbourhoods of $x \in S$:

$$U_x(\varepsilon,\lambda) = \{y: F_{x,y}(\varepsilon) > 1-\lambda\} \quad \varepsilon > 0, \quad \lambda \in (0,1) .$$

A triplet $(S,F,t)$ is a Menger space iff $(S,F)$ is a probabilistic metric space and $t$ is a T-norm such that for every $\varepsilon_1, \varepsilon_2 \in R^+$

$$F_{x,y}(\varepsilon_1+\varepsilon_2) \geq t(F_{x,z}(\varepsilon_1),\ F_{z,y}(\varepsilon_2))$$

for every $x,y,z \in S$.

A set $M \subset S$ is probabilistic bounded iff

$$\sup_{\varepsilon} D_M(\varepsilon) = 1$$

where

$$D_M(\varepsilon) = \sup_{\delta<\varepsilon} \inf_{x,y\in S} F_{x,y}(\delta)$$

is the probabilistic diameter of the set $M$.

If $f: S \to S$ and $x \in S$ then

$$O_f(x) = \{x, f(x), f^2(x), \dots\} .$$

DEFINITION 1. |1| *Let $(S,F)$ be a probabilistic metric space and $f: S \to S$. A point $x \in S$ is regular for $f$ iff* $\sup_{\varepsilon} D_{O_f(x)}(\varepsilon) = 1$.

DEFINITION 2. |1| *Let $(S,F)$ be a probabilistic metric space and $f: S \to S$. Two points $x,y \in S$ are asymptotic under $f$ iff*

$$F_{f^n(x),f^n(y)}(\varepsilon) \to 1 \quad \textit{for } n \to \infty , \quad \textit{for every} \quad \varepsilon > 0.$$

THEOREM 1. *Let $(S,F,t)$ be a Menger space with a continuous T-norm $t$ and let the mapping $f: S \to S$ be such that every point from $S$ be regular for $f$ and every pair of points from $S$ is asymptotic under $f$. Let $D_{O_f(x)} < D_{O_f(f(x))}$ whenever $D_{O_f(x)} < H$. If for some $x \in S$ the sequence $\{f^n(x)\}_{n\in N}$ has a convergent subsequence $\{f^{n_k}(x)\}_{n_k\in N}$ with a limit $a$ and if*

$$\sup_{\delta<t} \phi(\delta) = \phi(t) ,$$

*where*

$$\phi(\delta) = \inf_{x,y\in O_f f(a)} F_{x,y}(\delta) ,$$

*then the point $a$ is a unique fixed point of the mapping $f$.*

P r o o f. First we shall prove that from the conditions of the theorem if follows that

$$F_{a,f^m(a)}(\varepsilon) \geq D_{O_f(f(a))}(\varepsilon)$$

for all $m \in N$ and for all $\varepsilon > 0$. From the definition of a Menger space we have for all $\delta < \varepsilon$

$$F_{a,f^m(a)}(\varepsilon) \geq t(F_{a,f^{n_k}(x)}(\frac{\delta}{2}), F_{f^{n_k}(x),f^m(a)}(\varepsilon - \frac{\delta}{2})) \geq$$

$$\geq t(F_{a,f^{n_k}(x)}(\frac{\delta}{2}), t(F_{f^{n_k}(x),f^{n_k}(a)}(\frac{\delta}{2}), F_{f^{n_k}(a),f^m(a)}(\varepsilon-\delta))).$$

Also we have that

$$F_{f^{n_k}(a),f^m(a)}(\varepsilon-\delta) \geq \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta),$$

which means that

$$F_{a,f^m(a)}(\varepsilon) \geq t(F_{a,f^{n_k}(x)}(\frac{\delta}{2}), t(F_{f^{n_k}(x),f^{n_k}(a)}(\frac{\delta}{2}),$$

$$\inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))).$$

If $n_k \to \infty$, we have

$$\lim_{n_k\to\infty} F_{a,f^m(a)}(\varepsilon) \geq \lim_{n_k\to\infty} t(F_{a,f^{n_k}(x)}(\frac{\delta}{2}), t(F_{f^{n_k}(x),f^{n_k}(a)}(\frac{\delta}{2}),$$

$$, \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))).$$

From the preceding inequality we get

$$F_{a,f^m(a)}(\varepsilon) \geq t(H(\frac{\delta}{2}), t(H(\frac{\delta}{2}), \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))),$$

since

$$\lim_{n_k\to\infty} F_{a,f^{n_k}(x)}(\frac{\delta}{2}) = H(\frac{\delta}{2}) \quad (\lim_{n_k\to\infty} f^{n_k}(x) = a)$$

$$\lim_{n_k\to\infty} F_{f^{n_k}(x),f^{n_k}(a)}(\frac{\delta}{2}) = H(\frac{\delta}{2})$$

($x$ and $a$ are asymptotic points). Therefore, we have

$$F_{a,f^m(a)}(\varepsilon) \geq t(1, t(1, \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))) \geq$$

$$\geq t(1, t(1, \sup_{0<\delta<\varepsilon} \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))) = D_{O_f(f(a))}(\varepsilon).$$

So we have proved that

$$F_{a,f^m(a)}(\varepsilon) \geq D_{O_f(f(a))}(\varepsilon) .$$

Now we shall show that

$$D_{O_f(a)}(\varepsilon) = D_{O_f(f(a))}(\varepsilon) .$$

We have

$$D_{O_f(a)}(\varepsilon) = \sup_{t<\varepsilon} \inf_{n,k\in N\cup\{0\}} F_{f^n(a),f^k(a)}(t) =$$

$$\sup_{t<\varepsilon} (\min\{\inf_{n\in N} F_{a,f^n(a)}(t), \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t)\}) .$$

From the next two inequalities

$$F_{a,f^m(a)}(\varepsilon) \geq D_{O_f(f(a))}(\varepsilon) = \inf_{n,k\in N} F_{f^n(a),f^k(a)}(\varepsilon)$$

$$\inf_{n\in N} F_{a,f^n(a)}(t) \geq \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t) ,$$

it follows that

$$\min\{\inf_{n\in N} F_{a,f^n(a)}(t), \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t)\} =$$

$$= \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t)$$

and then we get

$$D_{O_f(a)}(\varepsilon) = \sup_{t<\varepsilon} \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t) = D_{O_f(f(a))}(\varepsilon) .$$

Since $D_{O_f(x)} < D_{O_f(f(x))}$ for all $x \in S$ whenever $D_{O_f(f(x))} < H$, it follows that

$$D_{O_f(a)} = H = D_{O_f(f(a))} .$$

Since

$$F_{a,f^m(a)}(\varepsilon) \geq D_{O_f(f(a))}(\varepsilon) = H(\varepsilon)$$

for all $m \in N$, and every $\varepsilon > 0$ for $m = 1$ we have that

$$F_{a,f(a)}(\varepsilon) = H(\varepsilon)$$

which means that

$$f(a) = a.$$

If we suppose that a point $b \in S$ is also a fixed point of the mapping f, then we have

$$F_{a,b}(\varepsilon) = F_{f^n(a),f^n(b)}(\varepsilon) \to H(\varepsilon)$$

for $n \to \infty$ (a,b are asymptotic points). From the last equality we get $a = b$.

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REZIME

### TEOREMA O NEPOKRETNOJ TAČKI U VEROVATNOSNIM METRIČKIM PROSTORIMA

U ovom radu je dokazana teorema o fiksnoj tački za preslikavanja u verovatnosnim metričkim prostorima. Ona predstavlja generalizaciju teoreme o fiksnoj tački preslikavanja u metričkim prostorima dokazanoj u |3|. Teorem glasi: Neka je $(S,F,t)$ Mengerov prostor sa neprekidnom T-normom t i neka je preslikavanje $f: S \to S$ takvo da je svaka tačka iz S regularna za f i svaki par tačaka iz S je asimptotski u odnosu na f. Neka je $D_{O_f(x)} < D_{O_f(f(x))}$ kadgod je $D_{O_f(x)} < H$. Ako za neko $x \in S$ postoji niz $\{f^n(x)\}_{n\in N}$ takav da konvergentni podniz $\{f^{n_k}(x)\}_{n_k\in N}$ ima graničnu vrednost a i ako je

$$\sup_{\delta<t} \phi(\delta) = \phi(t) \text{ , gde je}$$

$$\phi(\delta) = \inf_{x,y\in O_f(a)} F_{x,y}(\delta) \text{ ,}$$

tada je tačka a jedinstvena nepokretna tačka preslikavanja f.