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ON MULTIFUNCTIONS AND PARACOMPACTNESS

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ABSTRACT

In a paper |15| V.Popa introduced new classes of multifunctions called upper almost continuous (u.a.c.), lower almost continuous (l.a.c.) and almost continuous (a.c.) which are generalizations of upper semicontinuous (u.a.c.), lower semicontinuous (l.s.c.) and continuous multifunctions respectively.

The purpose of the present paper is to investigate some properties of these multifunctions and to obtain a new characterization for paracompact spaces.

1. PRELIMINARIES

DEFINITION 1.1. *A subset of a space is said to be regularly open iff it is the interior of some closed set or equivalently iff it is the interior of its own closure. A set is said to be regularly closed iff it is the closure of some open set or equivalently iff it is the closure if its own interior,* |3|.

A subset is regularly open iff its complement is regularly closed.

DEFINITION 1.2. *A subset* A *of a space* X *is* α*-nearly compact (*N*-closed) iff every* X*-regularly open cover of* A *has a finite subcovering,* |17|.

DEFINITION 1.3. *Let* X *be a topological space and* A *a subset of* X*. The set* A *is* α*-paracompact iff every* X*-open* *cover of* A *has an* X-*open* X-*locally finite refinement which covers* A, |18|.

DEFINITION 1.4. *A space* X *is paracompact iff every open cover has an open locally finite refinement,* |2|.

In this paper we shall observe multifunctions $F : X \to Y$ such that for each point $x \in X$, $F(x) \neq \emptyset$

DEFINITION 1.5. *The inferior inverse (or briefly, the inverse) of the multifunction* $F: X \to Y$ *is the multifunction denoted by* $F^{-}: Y \to X$ *and given* $F^{-}(y) = \{x \in X : y \in F(x)\}$, |1|.

For a multifunction $F : X \to Y$, *we shall denote by* $F^{+}(G)$ *and* $F^{-}(G)$ *the upper and lower inverse of the set* $G \subset Y$ *as in* |15|, *and thus*

$$F^{+}(G) = \{x \in X : F(x) \subset G\}; \quad F^{-}(G) = \{x \in X : F(x) \cap G \neq \emptyset\} .$$

DEFINITION 1.6. a) *The multifunction* $F : X \to Y$ *is upper semicontinuous (u.s.c.) (upper almost continuous, u.a.c.) in* $x_o \in X$ *iff for every open set* $V \subset Y$ *containing* $F(x_o)$ *there exists an open set* $U \subset X$ *containing* x_o *such that* $F(x) \subset V$ $(F(x) \subset \alpha(V) = \bar{V}^{o})$ *for each* $x \in U$, |1| (|15|).

b) *The multifunction* $F : X \to Y$ *is lower semicontinuous (l.s.c.) (lower almost continuous, l.a.c.) in* $x_o \in X$ *iff for every open set* $V \subset Y$ *with* $V \cap F(x_o) \neq \emptyset$ *there exists an open set* U *containing* x_o *such that* $F(x) \cap V \neq \emptyset$ $(F(x) \cap \alpha(V) \neq \emptyset)$ *for each* $x \in U$, |1| (|15|).

c) *The multifunction* $F : X \to Y$ *is continous (almost continuous) in* x_o *iff it is both u.s.c. and l.s.c. (u.a.c. and l.a.c.) in* x_o, |1| (|15|).

d) *The miltifunction* $F : X \to Y$ *is u.s.c., l.s.c. and continuous (u.a.c., l.a.c. and almost continuous) iff it has this property in each point* $x \in X$, |1| (|15|).

e) *The multifunction* $F : X \to Y$ *is closed (almost closed) iff for any closed (regularly closed) set* $A \subset X$, $F(A) = \cup\{F(x) : x \in A\}$ *is closed in* Y, |14|.

f) *The multifunction* $F : X \to Y$ *is open (almost open) iff for any open (regularly open) set* $A \subset X$, $F(A)$ *is open in* Y, |14|.

THEOREM 1.1. *For a multifunction* $F : X \to Y$ *the following are equivalent:*

a) F *is u.s.c. (l.s.c.) ,*

b) $F^+(G)$ *is open (closed) for each open (closed) set* $G \subset Y$,

c) $F^-(G)$ *is closed (open) for each closed (open) set* $G \subset Y$, |1|.

THEOREM 1.2. *For a multifunction* $F : X \to Y$ *the following are equivalent:*

a) F *is u.a.c. (l.a.c.) ;*

b) $F^+(G)$ *is open (closed) for each regularly open (regularly closed) set* $G \subset Y$ *;*

c) $F^-(G)$ *is closed (open) for each regularly closed (regularly open) set* $G \subset Y$, |15|.

DEFINITION 1.7. *A function* $f : X \to Y$ *is said to be almost continuous iff for each point* $x \in X$ *and each open neighbourhood* V *of* $f(x)$ *in* Y*, there exists an open neighbourhood* U *of* x *in* X *such that* $f(U) \subset \alpha(V)$, |16|.

A function is almost continuous iff the inverse image of every regularly open set is open, |16|.

DEFINITION 1.8. *A function* $f : X \to Y$ *is said to be almost closed (almost open) iff for every regularly closed (regularly open) set* F *of* X , $f(F)$ *is closed (open) in* Y, |16|.

2. SOME CHARACTERIZATIONS OF MULTIFUNCTIONS AND PARACOMPACTNESS

LEMMA 2.1. *If a multifunction* $F : X \to Y$ *is almost continuous and almost open, then*

a) *for each regularly open set* V *of* Y, $F^+(V)$ *is regularly open ;*

b) *for each regularly closed set* B *of* Y, $F^-(B)$ *is regularly closed.*

P r o o f. a) Let V be any regularly open set of Y. Since F is almost continuous $F^+(V)$ is open, hence we have

$F^{+}(V) \subset \alpha(F^{+}(V))$. On the other hand, since F is almost continuous and $\bar{V}$ is regularly closed, $F^{+}(\bar{V})$ is closed, hence we have $\alpha(F^{+}(V)) \subset F^{+}(V) \subset F^{+}(\bar{V})$. Since $\alpha(F^{+}(V))$ is regularly open and F is almost open, then $F(\alpha(F^{+}(V)))$ is open. Hence we have $F(\alpha(F^{+}(V))) \subset F(F^{+}(\bar{V})) \subset \bar{V}$, i.e. $F(\alpha(F^{+}(V))) \subset \alpha(V) = V$. Hence we have $\alpha(F^{+}(V)) \subset F^{+}(V)$. Hence $\alpha(F^{+}(V)) = F^{+}(V)$, i.e. $F^{+}(V)$ is regularly open.

b) Let B be any regularly closed subset of Y. Then $Y \setminus B$ is regularly open and $F^{-}(B) = F^{-}(Y \setminus (Y \setminus B)) = X \setminus F^{+}(Y \setminus B)$. Since $F^{+}(Y \setminus B)$ is regularly open, then $F^{-}(B)$ is regularly closed.

COROLLARY 2.1. *(|10|) If a mapping* $f : X \to Y$ *is almost continuous and almost open, then*

a) *for each regularly open set* V *of* Y, $f^{-1}(V)$ *is regularly open,*

b) *for each regularly closed set* B *of* Y, $f^{-1}(B)$ *is regularly closed.*

LEMMA 2.2. *A surjective multifunction* $F : X \to Y$ *is almost closed iff for every subset* $S \subset Y$ *and any regularly open set* U *of a space* X *containing* $F^{-}(S)$, *there exists an open set* V *in* Y *such that* $S \subset V$ *and* $F^{-}(V) \subset U$.

P r o o f. Let F be any almost closed multifunction of a space X onto a space Y. Let S be any set of Y and U be any regularly open set in X containing $F^{-}(S)$. Let $V = Y \setminus F(X \setminus U)$. Then, since $F^{-}(S) \subset U$, we have $S \subset V$. Since U is regularly open and F is almost closed, then V is open in Y. Now we have $F^{-}(V) = X \setminus F^{+}(F(X \setminus U)) \subset U$.

Now, let A be any regularly closed subset of X and $y \in Y \setminus F(A)$ be any point. Then we have $F^{-}(y) \subset F^{-}(Y \setminus F(A)) = X \setminus F^{+}(F(A)) \subset X \setminus A$. There exists an open set V in Y such that $y \in V$ and $F^{-}(V) \subset X \setminus A$. Thus we have $y \in V \subset Y \setminus F(A)$, hence $Y \setminus F(A)$ is open. Therefore, F(A) is closed, i.e. F is almost closed.

COROLLARLY 2.2. *(|10|) A surjective mapping* $f: X \to Y$ *is almost closed iff for any subset* $S \subset Y$ *and any regularly open set* U *of* X *containing* $f^{-1}(S)$, *there exists an open set* V *in* Y *such that* $S \subset V$ *and* $f^{-1}(V) \subset U$.

THEOREM 2.1. *If* X *is regular and multifunction* $F: X \to Y$ *is an almost closed surjection such that* $F^{-}(y)$ *is* α*-paracompact for each point* $y \in Y$, *then* F *is closed.*

P r o o f. Let A be any closed subset of X and let $y \in Y \setminus F(A)$. Since $F^{-}(y) \subset X \setminus A$ and X is regular, then for each point $x \in F^{-}(y)$ there exist open sets U_x and V_x such that $x \in U_x$, $A \subset V_x$, $U_x \cap V_x = \emptyset$. Now, $\mathcal{U} = \{U_x : x \in F^{-}(y)\}$ is an open covering of $F^{-}(y)$. Since $F^{-}(y)$ is α-paracompact, there exists an open X-locally finite family $\mathcal{W}$ which refines $\mathcal{U}$ and covers $F^{-}(y)$. For each $W \in \mathcal{W}$, there exists $U_{x_W} \in \mathcal{U}$ such that $W \subset U_{x_W}$. Since $U_{x_W} \cap V_{x_W} = \emptyset$, we have $\bar{W} \cap V_{x_W} = \emptyset$, i.e. $\bar{W} \cap A = \emptyset$. Let $V = \bigcup\{W : W \in \mathcal{W}\}$. Since $\bar{V} = \overline{\bigcup\{W : W \in \mathcal{W}\}} = \bigcup\{\bar{W} : W \in \mathcal{W}\}$ we have $\bar{V} \cap A = \emptyset$. Since $\overline{\alpha(V)} = \bar{V}$, we have $\overline{\alpha(V)} \cap A = \emptyset$. Therefore $\alpha(V)$ is a regularly open set containing $F^{-}(y)$ such that $F^{-}(y) \subset \alpha(V) \subset X \setminus A$. Since F is almost closed, then there exists an open set G in Y such that $y \in G$ and $F^{-}(G) \subset \alpha(V) \subset X \setminus A$. Therefore we have, $y \in G \subset Y \setminus F(A)$. Hence $Y \setminus F(A)$ is open i.e. $F(A)$ is closed. Hence F is closed.

COROLLARY 2.3. *(|6|) If* X *is regular and* $f: X \to Y$ *is an almost closed surjection such that* $f^{-1}(y)$ *is* α*-paracompact for each point* $y \in Y$, *then* f *is closed.*

LEMMA 2.3. *Let a multifunction* $F: X \to Y$ *be an almost closed surjection such that* $F^{-}(y)$ *is* α*-nearly compact for each point* $y \in Y$. *If* $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ *is an open locally finite family then* $F(\mathcal{U}) = \{F(U_\alpha) : \alpha \in I\}$ *is a locally finite family.*

P r o o f. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any open locally finite family in a space X. Let $y \in Y$ be any point in Y. Then for each point $x \in F^{-}(y)$ there exists an open neighbourhood $G(x)$ of x such that $G(x)$ interesects finitely many members of $\mathcal{U}$. Let $I(x)$ be a finite subset of I such that $G(x)\cap U_\alpha \neq \emptyset$, $\alpha \in I(x)$ and $G(x)\cap U_\alpha = \emptyset$, $\alpha \in I\setminus I(x)$.

The family $\{G(x): x \in F^-(y)\}$ is an X-open cover of $F^-(y)$. Since $F^-(y)$ is α-nearly compact, there exists a finite number of points $x_1, x_2, \dots, x_n$ in $F^-(y)$ such that $F^-(y) \subset \cup\{\alpha(G(x_i)) : i=1,2,\dots,n\}$. Now, let $G = \left(\cup\{\overline{(G(x_i)}: i=1,2,\dots,n\}\right)^o$. G is a regularly open set of X containing $F^-(y)$ such that $G\cap U_\alpha = \emptyset$, $\alpha \in I \setminus \bigcup_{i=1}^{n} I(x_i)$. If $G=X$, then a family $\mathcal{U}$ is finite, hence $F(\mathcal{U})$ is finite, i.e. locally finite. Let $G\neq X$. Since F is almost closed, then there exists an open set V of Y containing y such that $F^-(V)\subset G$. Thus we have $V\cap F(U_\alpha) = \emptyset$, for every $\alpha \in I\setminus \bigcup_{i=1}^{n} I(x_i)$. This implies that $F(\mathcal{U}) = \{F(U_\alpha): \alpha \in I\}$ is locally finite.

COROLLARY 2.4. *(|11|) Let* $f: X \to Y$ *be an almost closed surjection with N-closed point inverses. If* $\{U_\alpha: \alpha \in I\}$ *is a locally finite open cover of X, then* $\{f(U_\alpha): \alpha \in I\}$ *is a locally finite cover of Y.*

THEOREM 2.2. *If F is an almost closed u.s.c. and an open multifunction of a paracompact space X onto a space Y such that* $F^-(y)$ *is α-nearly compact for each point* $y \in Y$ *and* $F(x)$ *is α-paracompact for each point* $x \in X$*, then Y is paracompact.*

P r o o f. Let $\mathcal{U} = \{U_\alpha: \alpha \in I\}$ be any open covering of Y. Since for each point $x \in X$ the set $F(x)$ is α-paracompact and $\mathcal{U}$ is Y-open covering of $F(x)$, then there exists a Y-open Y-locally family $\{G_\lambda: \lambda \in I_x\}$ which refines $\mathcal{U}$ and is such that $F(x)\subset \cup\{G_\lambda: \lambda \in I_x\}$. Let $G_x = \cup\{G_\lambda: \lambda \in I_x\}$. G_x is an open set such that $F(x)\subset G_x$. Since F is u.s.c. then $\{F^+(G_x): x \in X\}$ is an open cover of X. Since X is paracompact, there exists an open locally finite refinement $\mathcal{V} = \{V_\beta: \beta \in J\}$ of $\{F^+(G_x): x \in X\}$. Since F is almost closed and $F^-(y)$ is α-nearly compact for each point $y \in Y$, then $F(\mathcal{V}) = \{F(V_\beta): \beta \in J\}$ is locally finite. Since F is open and $\mathcal{V}$ is an open locally finite covering of X it is obvious that $F(\mathcal{V})$ is an open locally finite covering of Y. For each $\beta \in J$ there exists an element $x_\beta \in X$ such that $V_\beta \subset F^+(G_{x_\beta})$, i.e. $F(V_\beta) \subset F(F^+(G_{x_\beta})) \subset G_{x_\beta}$. Let

$$V^* = \{F(V_\beta) \cap G_\lambda : \lambda \in I_{x_\beta},\ \beta \in J\} .$$

V^* is an open locally finite refinement of U, hence Y is paracompact.

COROLLARY 2.5. *(|13|) If* F *is a closed u.s.c. and an open multifunction of a paracompact space* X *onto a space* Y *such that* $F^-(y)$ *is compact for each point* $y \in Y$ *and* $F(x)$ *is compact for each point* $x \in X$*, then* Y *is paracompact.*

COROLLARY 2.6. *(|5|) If* $f : X \to Y$ *is an almost closed continuous and open mapping of a paracompact space* X *onto a space* Y *such that* $f^{-1}(y)$ *is* α*-nearly compact for each point* $y \in Y$*, then* Y *is paracompact.*

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REZIME

O VIŠEZNAČNIM PRESLIKAVANJIMA I PARAKOMPAKTNOSTI

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