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ON WONG'S CONJUCTURE

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ABSTRACT

In this paper we prove the existence of the differential equation

$$y'' + a(x)y = 0, \quad x \geq x_1$$

such that all the solutions are in $L^2[x_1,\infty)$ but at the same time they are not bounded.

J.S.W.Wong |1| gave the following conjucture: "If all solutions of the equation

(*) $\qquad y'' + a(x)y = 0, \quad x \geq a$

belong to $L^2[a,\infty)$ then it is Lagrange stable too".

The aim of this paper is to give a counterexample.

Because of the theorem, given by Wong and Patula |2|, which states that if (*) is nonoscillatory then it is a limit point too, our solutions have to be oscillatory functions.

Let

$$y_1(x) = e^{-\frac{x}{2}} \sin e^x + \bar{\delta}(x), \quad x \geq x_1 ,$$

where

$$\delta(x) = \frac{(x-x_n)^3 (x_{n+1}-x)^3}{(x_{n+1}-x_n)^6} \log_2 n \, \operatorname{sgn}(\sin e^x), \quad x \in [x_n, x_{n+1}) \equiv J_n$$

$$\bar{\delta}(x) = \begin{cases} 0, & x \notin J_{2^n} \\ \delta(x), & x \in J_{2^n} \end{cases}$$

and

$$x_n = \ln n\pi, \quad n \in N.$$

As $\max_{x \in J_n} |\delta(x)| = 2^{-6}\log_2 n$ it follows that $y_1(x)$ is unbounded for $x \in [x_1,\infty)$. The following calculation

$$\int_{x_1}^{\infty} \bar{\delta}^2(x)dx = \sum_{n=o}^{\infty} \int_{J_{2^n}} \delta^2(x)dx = \sum_{n=o}^{\infty} 2^{-12}n^2\ln(1+2^{-n}) \leq$$

$$\leq 2^{-12} \sum_{n=o}^{\infty} \frac{n^2}{2^n} < \infty$$

shows that $y_1(x) \in L^2[x_1,\infty)$.

For equation (*), the second linearly independent solution is given by

$$y^2(x) = y_1(x) \int^{x} \frac{dt}{y_1^2(t)}$$

which implies that

$$y_2^2(x) = y_1^2(x) \left(\int^{x} \frac{dt}{y_1^2(t)} \right)^2 \leq y_1^2(x) \int^{x} \frac{e^t dt}{\sin^2 e^t} = y_1^2(x)\operatorname{ctg}^2 e^x \leq$$

$$\leq 2(e^{-x}\sin^2 e^x + \bar{\delta}^2(x))\operatorname{ctg}^2 e^x = 2e^{-x}\cos^2 e^x + 2\bar{\delta}^2(x)\operatorname{ctg}^2 e^x.$$

Obviously $e^{-x}\cos^2 e^x \in L[x_1,\infty)$ and it remains to be proved that $\bar{\delta}^2(x)\operatorname{ctg}^2 e^x \in L[x_1,\infty)$.

Denote by x'_{2^n} and x''_{2^n}, $x'_{2^n} < x''_{2^n}$, the solutions of the equation $\operatorname{ctg}^2 e^x = 1$, $x \in J_{2^n}$. Then

$$\int_{x_1}^{\infty} \bar{\delta}^2(x)\operatorname{ctg}^2 e^x dx = \sum_{n=o}^{\infty} \int_{J_{2^n}} \bar{\delta}^2(x)\operatorname{ctg}^2 e^x dx =$$

(**)

$$= \sum_{n=o}^{\infty} \int_{x_{2^n}}^{x'_{2^n}} \delta^2(x)\operatorname{ctg}^2 e^x dx + \sum_{n=o}^{\infty} \int_{x'_{2^n}}^{x''_{2^n}} \delta^2(x)\operatorname{ctg}^2 e^x dx +$$

$$+ \sum_{n=o}^{\infty} \int_{x''_{2^n}}^{x_{2^n+1}} \delta^2(x)\operatorname{ctg}^2 e^x dx .$$

We shall prove that all the three series of (**) are convergent. It is obvious for the second one. For the first we have

$$\sum_{n=o}^{\infty}\int_{x_{2^n}}^{x'_{2^n}} \delta^2(x)\,ctg^2 e^x dx = \sum_{n=o}^{\infty}\int_{x_{2^n}}^{x'_{2^n}} (x-x_{2^n})^6 (x_{2^n+1}-x)^6 n^2 (x_{2^{n}+1} -$$

$$- x_{2^n})^{-12} ctg^2 e^x dx \le \sum_{n=o}^{\infty} (x_{2^n+1}-x_{2^n})^{-2} n^2 \int_{x_{2^n}}^{x'_{2^n}} \frac{(x-x_{2^n})^2}{\sin^2 e^x}\, dx < \sum_{n=o}^{\infty} \frac{n^2}{2^n} < \infty$$

because of

$$\frac{1}{2^n\pi} \le \left| \frac{x-x_{2^n}}{\sin e^x} \right| \le (x'_{2^n} - x_{2^n})\sqrt{2} < (x_{2^n+1} - x_{2^n}).$$

In a similar way we can prove that the third series is also convergent.

REFERENCES

|1| J.S.W.Wong, *Square integrable solutions of L^p perturbations of second order linear differential equations. Ordinary and partial differential equations p. 282-292. Lecture Notes in Math. Vol. 415. Springer Verlag, Berlin 1971.*

|2| W.T.Patula, J.S.W.Wong, *An L^p analogue of the Weyl alternative. Math. Ann. 197, p. 9-28 (1972).*

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REZIME

O JEDNOJ WONGOVOJ PRETPOSTAVCI

U radu je dokazana egzistencija diferencijalne jednačine

$$y'' + a(x)y = 0, \quad x \ge x_1$$

čija su sva rešenja iz $L^2[x_1,\infty)$ ali nisu i ograničena. Time je oborena pretpostavka J.S.W.Wonga.