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ON HERMITE POLYNOMIALS OF THE GAUSSIAN RANDOM PROCESS

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ABSTRACT

An explicit expression is defined to be a Hermite polynomial of degree n, n=1,2,..., in variables $\xi(t_1), \xi(t_2), \ldots, \xi(t_n) \in \{\xi(t),\ t \in T\}$, where $\{\xi(t),\ t \in T\}$ is a real Gaussian process. Some properties of these polynomials are investigated. Especially, $E^S H_n(\xi_1, \xi_2, \ldots, \xi_n) = H_n(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n)$, where $\hat{\xi}_k = E^S \xi_k$.

Let $\{\xi(t),\ t \in T\}$ be a real Gaussian random process on a probability space (Ω, F, P), with the expectation $E\xi(t) = 0,\ t \in T$ and covariance function $E\xi(t_i)\xi(t_j) = B(t_i, t_j),\ t_i, t_j \in T$. Let $\mathbf{H}(T)$ be Hilbert space of square integrable functions, measurable with respect to the σ-field $F(T)$, generated by $\{\xi(t),\ t \in T\}$, with the scalar product

$$(\phi, \psi) = E\phi\psi \ , \quad \phi, \psi \in \mathbf{H}(T) \ .$$

We take that $E\phi = 0,\ \phi \in \mathbf{H}(T)$.

In |4| a Hermite polynomial of degree n, in variables $\xi(t_1), \xi(t_2), \ldots, \xi(t_k),\ t_1, \ldots, t_k \in T$ is defined as a polynomial orthogonal to all polynomials of a degree less than n. It is shown that

$$\mathbf{H}(T) = \sum_{p=1}^{\infty} \oplus H_p(T) \ ,$$

where $H_p(T)$ is a linear closure spanned by Hermite polynomials of degree p, in variables $\xi(t),\ t \in T$. In |3|, explicit expressions for Hermite polynomials of degree 1, 2, and 3 are given.

In this paper an explicit expression is defined as a Hermite polynomial of degree n, ,n=1,2,... and some properties of these polynomials are investigated (Theorem 3, 4, 5).

A well known formula for the moments of the Gausiian n-dimensional random variable $(\xi(t_1),\xi(t_2),\ldots,\xi(t_n))$, where n is even, will be used

$$E\xi(t_1)\xi(t_2)\ldots\xi(t_n) = \Sigma\prod B(t_i,t_j) \quad (1)$$

where the sum ranges over all the possible partitions of the set $\{t_1,t_2,\ldots,t_n\}$ to pairs $\{t_i,t_j\}$ and the product ranges over all the pairs $\{t_i,t_j\}$ of a corresponding partition.

DEFINITION *A Hermite polynomial, of degree* n *of* n *variables, is*

$$H_n(\xi_1,\xi_2,\ldots,\xi_n) = \xi_1\xi_2\ldots\xi_n - \Sigma B_{i_1j_1}\alpha_{i_1j_1}(\xi_1,\ldots,\xi_n) + \quad (2)$$

$$+ \Sigma B_{i_1j_1}B_{i_2j_2}\alpha_{i_1j_1i_2j_2}(\xi_1,\ldots,\xi_n) + \ldots + (-1)^k \Sigma B_{i_1j_1}\ldots$$

$$\ldots B_{i_kj_k}\alpha_{i_1j_1\ldots i_kj_k}(\xi_1,\ldots,\xi_n) + \ldots + (-1)^{[\frac{n}{2}]}\Sigma\prod B_{ij},$$

$$\xi_i = \xi(t_i),\ B_{i_\nu j_\nu} = B(t_{i_\nu},t_{j_\nu}),\ i_\nu,j_\nu \in \{1,2,\ldots,n\}.$$

We take that $i_\nu < j_\nu$ which is not a restriction,

$$\alpha_{i_1j_1\ldots i_kj_k}(\xi_1,\ldots,\xi_n) = \prod_{i=1}^{n}\xi_i,\quad i\neq i_\nu,\quad i\neq j_\nu,\ \nu=1,2,\ldots,k,$$

$$k=1,2,\ldots,[\tfrac{n}{2}].$$

The first sum, in (2), ranges over all the possible choices of a pair (i_1,j_1) from the set 1,2,...,n. The k-th sum, k=2,3,... ..., $[\frac{n}{2}]-1$, in (2), ranges over all the possible choices of k pairs (i_1,j_1), $(i_2,j_2),\ldots,(i_k,j_k)$, from the set $\{1,2,\ldots,n\}$, where the order of pairs is not important. The last term in (2) is $\Sigma\prod B_{ij}$, where the sum ranges over all the possible partitions of the set $\{1,2,\ldots,n\}$ to pairs (i,j) and the product ranges over all the pairs of a corresponding partition. Notice that $\Sigma\prod B_{ij}$ is equal to $E\xi_1\xi_2\ldots\xi_n$, given in (1).

$\Sigma B_{i_1j_1}\alpha_{i_1j_1}(\xi_1,\dots,\xi_n$ has $\binom{n}{2}$ terms,

$\Sigma B_{i_1j_1}B_{i_2j_2}\alpha_{i_1j_1i_2j_2}(\xi_1,\dots,\xi_n)$ has $\frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \binom{n}{4}3!!$ terms,

$\Sigma B_{i_1j_1}\dots B_{i_kj_k}\alpha_{i_1j_1\dots i_kj_k}(\xi_1,\dots,\xi_n)$ has

$\frac{1}{k!}\binom{n}{2}\binom{n-2}{2}\dots\binom{n-2(k-1)}{2} = \binom{n}{2k}(2k-1)!!$ terms because there is

$\binom{n}{2}\binom{n-2}{2}\dots\binom{n-2(k-1)}{2} = A$ ways to choose k pairs from the set $\{1,2,\dots,n\}$, and as the order of pairs is not important, the exact number is $\frac{1}{k!}A$

$\Sigma\prod B_{ij}$ has $(n-1)!!$ terms.

The polynomial $H_n(\xi_1,\dots,\xi_n)$, for an n even, contains only elements of an even degree and the constant element.

For n odd, it contains only elements of an odd degree and the constant element is zero. In $H_n(\xi_1,\dots,\xi_n)$ we may have some equal variables. So we have a Hermite polynomial in one, two, there, e.t.c. variables of degree n. If $\xi_1 = \xi_2 = \dots = \xi_n$, we have the Hermite polynomial of one variable. For example, $H_1(\xi) = \xi$, $H_2(\xi) = \xi^2 - \sigma^2$, $H_3(\xi) = \xi^3 - 3\sigma^2\xi$, $H_4(\xi) = \xi^4 - 6\sigma^2\xi^2 + 3\sigma^4$, $H_5(\xi) = \xi^5 - 10\sigma^2\xi^3 + 15\sigma^4\xi$. In general

$$H_n(\xi) = \xi^n + \sum_{k=1}^{[\frac{n}{2}]}(-1)^k(2k-1)!!\binom{n}{2k}\sigma^{2k}\xi^{n-2k},$$

where $\xi : N(0,\sigma^2)$.

If $\xi(t)$, $t \in T$ are independent random variables, then $H_n(\xi_1,\dots,\xi_k) = H_{n_1}(\xi_1)H_{n_2}(\xi_2)\dots H_{n_k}(\xi_k)$, for $k \le n$, $n = n_1+n_2+\dots+n_k$. $H_{n_i}(\xi_i)$, $i=1,2,\dots,k$, are the Hermite polynomials of one variable, which is a well known definition of the Hermite polynomial of degree n, in k variables, see for instance |1|.

THEOREM 1.

$$E\,H_n(\xi_1,\dots,\xi_n) = 0.$$

P r o o f. The statement follows immediately for n odd, because all odd moments of the Gaussian n-dimensional random variable $(\xi_1,\dots,\xi_n)$ are zero. Let $n=2m$, $m=1,2,\dots$. Then

$$\text{(3)}\quad EH_n(\xi_1,\dots,\xi_n) = E\xi_1\xi_2\dots\xi_n - \Sigma B_{i_1j_1}E\alpha_{i_1j_1}(\xi_1,\dots,\xi_n)+\dots+ \\ +(-1)^k\,\Sigma B_{i_1j_1}\dots B_{i_kj_k}E\alpha_{i_1j_1\dots i_kj_k}(\xi_1,\dots,\xi_n)+\dots+ \\ +(-1)^m\;\Sigma\prod B_{ij}\,.$$

Let us notice an arbitrary term $B_{i_{1o}j_{1o}}B_{i_{2o}j_{2o}}\dots B_{i_{mo}j_{mo}}$ in the first member $E\xi_1\dots\xi_n$ of (3). This term occurs in the second member of (3) exactly $m=n/2$ times, because $B_{i_1j_1}$ can be any one of $B_{i_{1o}j_{1o}},\dots,B_{i_{2o}j_{2o}},\dots,B_{i_{mo}j_{mo}}$. It occurs $\binom{m}{k}$ times in the k-th member of (3), $k=2,\dots,m-1$, because $B_{i_1j_1},B_{i_2j_2},\dots,B_{i_kj_k}$ can be any k elements from $B_{i_{1o}j_{1o}},B_{i_{2o}j_{2o}},\dots,B_{i_{mo}j_{mo}}$. In the last member $\Sigma\prod B_{ij}$ it occurs only once. According this, the coefficient of $B_{i_{1o}j_{1o}}B_{i_{2o}j_{2o}}\dots B_{i_{mo}j_{mo}}$ in (3) is

$$1-m+\binom{m}{2}+\dots+(-1)^k\binom{m}{k}+\dots+(-1)^m\binom{m}{m}=0.$$

Since the product $B_{i_{1o}j_{1o}}\dots B_{i_{mo}j_{mo}}$ was arbitrary, the same is valid for any other term in $E\xi_1\dots\xi_n$.

THEOREM 2. *The Hermite polynomial* $H_n(\xi_1,\dots,\xi_n)$ *is orthogonal to every polynomial in* H(T) *of degree* k, $k<n$.

P r o o f. It is sufficient to see that

$$E\,H_n(\xi_1,\dots,\xi_n)\eta_1\eta_2\dots\eta_k=0,$$

where $\eta_i=\eta(t_i)\in\{\xi(t),\ t\in T\}$, $i=1,2,\dots,k$, $k<n$. If n is even and k odd, or conversely, the statement immediately follows, because, after multiplication, we get a polynomial with all the elements of odd degree, so, its expectation is zero. Let n and k be both odd, or both even. We shall show the case $n=2m$, $k=2q$, $m,q\in N$.

The proof is analogous when n and k are both odd. Denote

$$B(i_s,i_p)=E\xi_{i_s}\xi_{i_p},\quad i_s,i_p\in\{1,2,\dots,n\},$$

$$B(i_s,j_p)=E\xi_{i_s}\eta_{j_p},\quad i_s\in\{1,2,\dots,n\},\ j_p\in\{1,2,\dots,k\},$$

$$B(j_s,j_p)=E\eta_{j_s}\eta_{j_s},\quad j_s,j_p\in\{1,2,\dots,k\}.$$

The terms in $E\xi_1\dots\xi_n\eta_1\dots\eta_k$ have the form

(4) $$B(i_1,i_2)\dots B(i_{\nu-1},i_\nu)B(i_{\nu+1},j_1)\dots B(i_n,j_\mu)B(j_{\mu+1},j_{\mu+2})\dots$$

$$\dots B(j_{k-1},j_k),\ \nu\in\{2,4,\dots,n\},\ \mu=n-\nu,\ \mu\le k.$$

For $\nu=n$, $(\mu=0)$ we have

$$B(i_1,i_2)\dots B(i_{n-1},i_n)B(j_1,j_2)\dots B(j_{k-1},j_k).$$

For $\nu=n-k$, $(\mu=k)$ we have

$$B(i_1,i_2)\dots B(i_{n-k-1},i_{n-k})B(i_{n-k+1},j_1)\dots B(i_n,j_k).$$

Let us notice an arbitrary term in $E\xi_1\dots\xi_n\eta_1\dots\eta_k$ as it is given in (4). In $\Sigma B_{i_1j_1}E\alpha_{i_1j_1}(\xi_1,\dots,\xi_n)\eta_1\dots\eta_k$ the term (4) occurs $\binom{\nu/2}{1}=\nu/2$ times, because $B_{i_1j_1}$ can be any one of the $B(i_1,i_2),\dots,B(i_{\nu-1},i_\nu)$. In $\Sigma B_{i_1j_1}\dots B_{i_rj_r}E\alpha_{i_1j_1\dots i_rj_r}(\xi_1,\dots,\xi_n)\eta_1\dots\eta_k$, for $r\le\nu/2$, the term (4) occurs $\binom{\nu/2}{r}$ times, and for $r>\nu/2$ it does not occur. We have that in $EH_n(\xi_1,\dots,\xi_n)\eta_1\dots\eta_k$ the term (4) occurs

$$1-\binom{\nu/2}{1}+\dots+(-1)^r\binom{\nu/2}{r}+\dots+(-1)^{\nu/2}\binom{\nu/2}{\nu/2}=0$$

times. Since the term (4) is arbitrary, this is valid for any other term in $E\xi_1\dots\xi_n\eta_1\dots\eta_k$.

REMARK 1. Let $S_1,S_2\subseteq T$ be arbitrary and $H_p(S_1),H_q(S_2)$ the corresponding linear closures, spanned by the Hermite polynomial of degree p, in the variables $\xi(t_1),\dots,\xi(t_k)$, $t_1,\dots,t_k\in S_1$ and of degree q in the variables $\xi(t_1),\dots,\xi(t_j)$, $t_1,\dots,t_j\in S_2$. In |4| it is shown that

$$H_p(S_1) \perp H_q(S_2) \ , \quad p \neq q.$$

The proof follows also from Theorem 2., because $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_k$ are the arbitrary elements from $\{\xi(t),\ t\in T\}$.

THEOREM 3. *The Hermite polynomial* $H_n(\xi_1,\dots,\xi_n)$, n=2, 3,... *satisfies the partial differential equations*

$$\frac{\partial}{\partial\xi_k} H_n(\xi_1,\dots,\xi_n) = H_{n-1}(\xi_1,\dots,\xi_{k-1},\xi_{k+1},\dots,\xi_n),$$
$$k = 1,2,\dots,n \ .$$

P r o o f. Denote

$$\sum_{i_1,j_1,\dots,i_r,j_r\in\{1,\dots,n\}} B_{i_1j_1}\cdots B_{i_rj_r}\alpha_{i_1j_1\dots i_rj_r}(\xi_1,\dots,\xi_n)$$

by

$$\sum_{1,2,\dots,n} B_{i_1j_1}\cdots B_{i_rj_r}\alpha_{i_1j_1\dots i_rj_r}(\xi_1,\dots,\xi_n) \ .$$

$$\frac{\partial}{\partial\xi_1} H_n(\xi_1,\dots,\xi_n) = \xi_2\cdots\xi_n - \frac{\partial}{\partial\xi_1}\sum B_{i_1j_1}\alpha_{i_1j_1}(\xi_1,\dots,\xi_n) +$$
$$+\dots+(-1)^r \frac{\partial}{\partial\xi_1}\sum B_{i_1j_1}\cdots B_{i_rj_r}\alpha_{i_1j_1\dots i_rj_r}(\xi_1,\dots,\xi_n) +$$
$$+ \dots + (-1)^{\left[\frac{n}{2}\right]} \frac{\partial}{\partial\xi_1} \sum\prod B_{ij} \ .$$

$$\frac{\partial}{\partial\xi_1}\sum_{1,\dots,n} B_{i_1j_1}\alpha_{i_1j_1}(\xi_1,\dots,\xi_n) = \sum_{2,\dots,n} B_{i_1j_1}\alpha_{i_1j_1}(\xi_2,\dots,\xi_n) \ .$$

The sum on the right hand side ranges over all the choices of a pair (i_1,j_1) from the set $\{2,\dots,n\}$, because after differentiation all the terms with coefficient B_{1j}, $j\in\{2,\dots,n\}$ disappear, namely, the corresponding products $\alpha_{1j}(\xi_1,\dots,\xi_n)$ do not contain ξ_1. In the rest of the terms $B_{ij}\alpha_{ij}(\xi_1,\dots,\xi_n)$, (i,j) are chosen from the set $\{2,\dots,n\}$ and

$$\frac{\partial}{\partial\xi_1}\alpha_{ij}(\xi_1,\dots,\xi_n) = \alpha_{ij}(\xi_2,\dots,\xi_n) \ .$$

$$\frac{\partial}{\partial \xi_1} \sum_{1,\dots,n} B_{i_1j_1}\dots B_{i_rj_r}\alpha_{i_1j_1,\dots i_rj_r}(\xi_1,\dots,\xi_n) =$$

$$\sum_{2,\dots,n} B_{i_1j_1} \dots B_{i_rj_r}\alpha_{i_1j_1\dots i_rj_r}(\xi_2,\dots,\xi_n)$$

The sum on the right hand side renges over all the choices of r pairs $(i_1,j_1),\dots,(i_r,j_r)$, from the set $\{2,\dots,n\}$, because all the terms which do not contain ξ_1 in $\alpha_{i_1j_1\dots i_rj_r}(\xi_1,\dots,\xi_n)$ disappear. In the other terms, the indexes are chosen from the set $\{2,\dots,n\}$, and

$$\frac{\partial}{\partial \xi_1} \alpha_{i_1j_1\dots i_rj_r}(\xi_1,\dots,\xi_n) = \alpha_{i_1j_1\dots i_rj_r}(\xi_2,\dots,\xi_n).$$

This is independent of r, so it is valid for every term in $H_n(\xi_1,\dots,\xi_n)$. The same is true if we consider any of $\xi_2,\dots,\xi_n$ instead of ξ_1.

THEOREM 4. *The polynomials* $P_n(\xi_1,\dots,\xi_n)$, $n \geq 2$, $P_1(\xi_1) = \xi_1$, *which satisfy the partial differential equations*

$$\frac{\partial}{\partial \xi_k} P_n(\xi_1,\dots,\xi_n) = P_{n-1}(\xi_1,\dots,\xi_{k-1},\xi_{k+1},\dots,\xi_n)$$

$$k=1,2,\dots,n, \quad \xi_i = \xi(t_i), \quad t_i \in T, \quad i=1,2,\dots,n$$

with the initial condition $EP_n(\xi_1,\dots,\xi_n) = 0$ *are Hermite polynomials* $H_n(\xi_1,\dots,\xi_n)$, $n=1,2,\dots$.

Before the proof let us consider the following examples

1. $P_1(\xi_1) = \xi_1 = H_1(\xi_1)$

2. $\frac{\partial}{\partial \xi_1} P_2(\xi_1,\xi_2) = P_1(\xi_2) = \xi_2$, so $P_2(\xi_1,\xi_2) = \xi_1\xi_2 + \phi(\xi_2)$.

$$\frac{\partial}{\partial \xi_2} P_2(\xi_1,\xi_2) = P_1(\xi_1) = \xi_1 = \xi_1 + \phi'(\xi_2), \text{ so } \phi'(\xi_2) = 0 ,$$

and $P_2(\xi_1,\xi_2) = \xi_1\xi_2 + c$. The constant c is evaluated from the condition $EP_2(\xi_1,\xi_2) = 0$, and it is equal to $-E\xi_1\xi_2 = -B_{12}$, so finally

$$P_2(\xi_1,\xi_2)=\xi_1\xi_2-B_{12}=H_2(\xi_1,\xi_2).$$

3. $\frac{\partial}{\partial\xi_1}P_3(\xi_1,\xi_2,\xi_3)=\xi_2\xi_3-B_{23}$, so, $P_3(\xi_1,\xi_2,\xi_3)=\xi_1\xi_2\xi_3-$ $-B_{23}\xi_1+\phi_1(\xi_2,\xi_3)$

$\frac{\partial}{\partial\xi_2}P_3(\xi_1,\xi_2,\xi_3)=\xi_1\xi_3+\frac{\partial}{\partial\xi_2}\phi_1(\xi_2,\xi_3)$, so, $\phi_1(\xi_1,\xi_2)=$ $=-B_{13}\xi_2+\phi_2(\xi_3)$, and $P_3(\xi_1,\xi_2,\xi_3)=\xi_1\xi_2\xi_3-B_{23}\xi_1-B_{13}\xi_2+\phi_2(\xi_3)$.

$\frac{\partial}{\partial\xi_3}P_3(\xi_1,\xi_2,\xi_3)=\xi_1\xi_2-B_{12}=\xi_1\xi_2+\frac{\partial}{\partial\xi_3}\phi(\xi_3)$, so, $\phi_2(\xi_3)=$ $=-B_{12}\xi_3+c$.

From $EP_3(\xi_1,\xi_2,\xi_3)=0$, it follows that $c=0$ and finally

$$P_3(\xi_1,\xi_2,\xi_3)=\xi_1\xi_2\xi_3-B_{23}\xi_1-B_{13}\xi_2-B_{12}\xi_3=H_3(\xi_1,\xi_2,\xi_3).$$

P r o o f. We know that $P_1(\xi_1)=\xi_1=H_1(\xi_1)$ and suppose that

$$P_{n-1}(\xi_1,\ldots,\xi_{n-1})=H_{n-1}(\xi_1,\ldots,\xi_{n-1}).$$

Then

$\frac{\partial}{\partial\xi_1}P_n(\xi_1,\ldots,\xi_n)=\frac{\partial}{\partial\xi_1}P_n=P_{n-1}(\xi_2,\ldots,\xi_n)=H_{n-1}(\xi_2,\ldots,\xi_n)=$

$=\xi_2\xi_3\cdots\xi_n-\sum_{2,\ldots,n}B_{ij}\alpha_{ij}(\xi_2,\ldots,\xi_n)+\ldots$. It follows that

$$P_n=\xi_1\xi_2\cdots\xi_n-\sum_{2,\ldots,n}B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n)\xi_1+\ldots+\phi_1(\xi_2,\ldots,\xi_n).$$

$\frac{\partial}{\partial\xi_2}P_n=\xi_1\xi_3\cdots\xi_n-\sum_{3,\ldots,n}B_{ij}\alpha_{ij}(\xi_3,\ldots,\xi_n)\xi_1+\ldots+\frac{\partial}{\partial\xi_2}\phi_1(\xi_2,$ $\ldots,\xi_n)=P_{n-1}(\xi_1,\xi_3,\ldots,\xi_n)=H_{n-1}(\xi_1,\xi_3,\ldots,\xi_n)=$

$=\xi_1\xi_3\cdots\xi_n-\sum_{1,3,\ldots,n}B_{ij}\alpha_{ij}(\xi_1,\xi_3,\ldots,\xi_n)+\ldots$, whence

$\frac{\partial}{\partial\xi_2}\phi_1(\xi_2,\ldots,\xi_n)=\sum_{j=3}^{n}B_{1j}\alpha_{1j}(\xi_1,\xi_3,\ldots,\xi_n)+\ldots$ and

$$P_n=\xi_1\xi_2\cdots\xi_n-\sum_{2,\ldots,n}B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n)-\sum_{j=3}^{n}B_{1j}\alpha_{1j}(\xi_1,\ldots,\xi_n)+\phi_2(\xi_3,\ldots,\xi_n)$$

$$\frac{\partial}{\partial\xi_3} P_n = \xi_1\xi_2\xi_3\cdots\xi_n - \sum_{2,4,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\xi_2,\xi_4,\ldots,\xi_n) -$$

$$- \sum_{j=4}^{n} B_{1j}\alpha_{1j}(\xi_1,\xi_2,\xi_4,\ldots,\xi_n) + \ldots + \frac{\partial}{\partial\xi_3}\phi_2(\xi_3,\ldots,\xi_n) =$$

$$= \xi_1\xi_2\xi_4\cdots\xi_n - \sum_{1,2,4,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\xi_2,\xi_4,\ldots,\xi_n) + \ldots,$$

whence

$$\frac{\partial}{\partial\xi_n}\phi_2(\xi_3,\ldots,\xi_n) = -B_{12}\alpha_{12}(\xi_1,\xi_2,\xi_4,\ldots,\xi_n) + \ldots, \text{ so}$$

$$P_n(\xi_1,\ldots,\xi_n) = \xi_1\cdots\xi_n - \sum_{2,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n) -$$

$$- \sum_{j=3}^{n} B_{1j}\alpha_{1j}(\xi_1,\ldots,\xi_n) - B_{12}\alpha_{12}(\xi_1,\ldots,\xi_n) + \ldots + \phi_3(\xi_4,\ldots,\xi_n) =$$

$$= \xi_1\cdots\xi_n - \sum_{1,2,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n) + \ldots + \phi_3(\xi_4,\ldots,\xi_n).$$

In the same way we get the third and the other elements in $P_n(\xi_1,\ldots,\xi_n) = H_n(\xi_1,\ldots,\xi_n)$.

Let $S \subset T$. Denote $E^S\xi(t) = E(\xi(t)\,|\,F(S))$. $t \in T$. Where $F(S)$ is the σ-algebra generated by $\{\xi(t),\ t \in S\}$.

In |2| it is shown that the conditional expectation, with respect to $F(S)$, of a polynomial of degree n, in variables $\xi(t_1)$, $\ldots,\xi(t_k)$, $t_1,\ldots,t_k \in T$, is a polynomial of degree n, in $\hat{\xi}(t_1),\ldots,\hat{\xi}(t_k)$, $t_1,\ldots,t_k \in T$, where $\hat{\xi}(t) = E^S\xi(t)$, $t \in T$.

THEOREM 5. $E^S H_n(\xi_1,\ldots,\xi_n) = H_n(\hat{\xi}_1,\ldots,\hat{\xi}_n)$.

P r o o f. We know that

$E^S H_n(\xi_1,\ldots,\xi_n) = P_n(\hat{\xi}_1,\ldots,\hat{\xi}_n)$ and

$E^S H_1(\xi_1) = E^S\xi_1 = \hat{\xi}_1$.

The conditional distribution of $(\xi_1,\xi_2,\ldots,\xi_n)$, given $F(S)$, is a Gaussian with the mean vector $(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n)$ and covariance matrix $B = \|b_{ij}\|$, where $b_{ij} = E^S(\xi_i-\hat{\xi}_i)(\xi_j-\hat{\xi}_j)$ are constant, because $\xi_k-\hat{\xi}_k$ is independent of $F(S)$. Let $\phi(x_1-\hat{\xi}_1, x_2-\hat{\xi}_2,\ldots x_n-\hat{\xi}_n\,|\,F(S))$ be the conditional density of $(\xi_1,\xi_2,\ldots,\xi_n)$. So

$$E^S H_n(\xi_1,\dots,\xi_n) = \int_{R^n} H_n(x_1,\dots,x_n)\phi(x_1-\hat{\xi}_1,\dots,x_n-\hat{\xi}_n \mid F(S))dx_1 \dots dx_k \dots dx_n$$

and

$$\frac{\partial}{\partial\hat{\xi}_k} E^S H_n = \int_{R^n} \frac{\partial}{\partial\hat{\xi}_k} H_n(y_1+\hat{\xi}_1,\dots,y_n+\hat{\xi}_n)\phi(y_1,\dots,y_n \mid F(S))dy_1 \dots dy_k \dots dy_n = \int_{R^n} H_{n-1}(y_1+\hat{\xi}_1,\dots,y_{k-1}+\hat{\xi}_{k-1},y_{k+1}+\hat{\xi}_{k+1},\dots,y_n+\hat{\xi}_n)\phi(y_1,\dots,y_n \mid F(S))dy_1\dots dy_n =$$

$$= \int_{R_{n-1}} H_{n-1}(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\phi(x_1-\hat{\xi}_1,\dots,x_{k-1}-\hat{\xi}_{k-1},x_{k+1}-\hat{\xi}_{k+1},\dots,x_n-\hat{\xi}_n \mid F(S))dx_1\dots dx_{k-1}dx_{k+1}\dots dx_n = E^S H_{n-1}(\xi_1,\dots,\xi_{k-1},\xi_{k+1},\dots,\xi_n) .$$

It follows that the polynomials $E^S H_n(\xi_1,\dots,\xi_n) = P_n(\hat{\xi}_1,\dots,\hat{\xi}_n)$ satisfy the partial differential equation

$$\frac{\partial}{\partial\hat{\xi}_k} P_n(\hat{\xi}_1,\dots,\hat{\xi}_n) = P_{n-1}(\hat{\xi}_1,\dots,\hat{\xi}_{k-1},\hat{\xi}_{k+1},\dots,\hat{\xi}_n), \quad n \geq 2,$$

and since

$$EP_n(\hat{\xi}_1,\dots,\hat{\xi}_n) = EE^S H_n(\xi_1,\dots,\xi_n) = 0,$$

from Theorem 2. we have

$$E^S H_n(\xi_1,\dots,\xi_n) = P_n(\hat{\xi}_1,\dots,\hat{\xi}_n) = H_n(\hat{\xi}_1,\dots,\hat{\xi}_n) .$$

REMARK 2. In |4| it is shown that for $\eta \in H_p(T)$, $E^S\eta \in H_p(S)$. The same result follows immediately from Theorem 5.

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REZIME

O ERMITSKIM POLINOMIMA GAUSOVOG SLUČAJNOG PROCESA

U radu se daje eksplicitan izraz za Ermitske polinome stepena n, n = 1,2,..., i ispituju se neke osobine ovih polinoma (Teoreme 3., 4., 5.) .