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ON SPECTRAL TYPE OF NONLINEAR AND NONANTICIPATIVE TRANSFORMATION OF THE WIENER PROCESS

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ABSTRACT

Let $H(W)$ be a Hilbert space of the square integrable functions of the Wiener proces $W(T), t \geq 0$. It is shown that there exists a process $\eta(t) = T\{W(u),\ 0 \leq u \leq t\},\ t \geq 0$ which has any given spectral type.

Let $\{E_t,\ t \geq t_o\}$ be a resolution of the identity in a separable Hilbert space H. According to the well-known Stone theorem the spectral type

(1) $$R_1(t) \geq R_2(t) \geq \ldots \geq R_N(t), \quad t \geq t_o$$

(the spectral multiplicity N may be $+\infty$) of $\{E_t\}$ is the complete system of the unitary invariants of $\{E_t\}$.

The Stone theorem is applied to a time-domain analysis of stochastic processes in a classical paper |1| in the following way: Let the second order process $\{\xi(t),\ t \geq t_o\}$ be continuous and purely nondeterministic. Let $H^{(1)}(\xi)$ $(H_\xi^{(1)}(t))$ be a Hilbert space - the linear closure over $\xi(t),\ t \geq t_o$ $(\xi(u),\ t_o \leq u \leq t)$. The scalar product is $(\xi_1, \xi_2) = E\xi_1\xi_2, \xi_i \in H^{(1)}(\xi)$, E is the expectation. So the family $\{P_t,\ t \geq t_o\}$ of the projection operators P_t onto $H_t^{(1)}(\xi)$ is the resolution of the identity.

Then there exist N and the mutually orthogonal wide sense martingals $\{\zeta_n(t),\ t \geq t_o\}$, $n = \overline{1,N}$ such that

(2) 1. $$H_t^{(1)}(\xi) = \sum_{n=1}^{N} \oplus H_t^{(1)}(\zeta_n),\quad t \geq t_o ,$$

2. The measure dF_{ζ_n} with the distribution function $F_{\zeta_n}(t) = \|\zeta_n(t)\|^2$, $t \geq t_o$, belongs to the class of the equivalent measure $R_n^{\xi}(t)$. The spectral type of $\{\xi(t)\}$

$$R_1^{\xi}(t) \geq R_2^{\xi}(t) \geq \dots \geq R_N^{\xi}(t),\quad t \geq t_o$$

is the spectral type of $\{P_t\}$.

The main result of |1| is that for the arbitrary spectral type (1) there exists a continuous process $\{\xi(t),\ t \geq t_o\}$ with this spectral type.

Let $\{W(t),\ t \geq t_o\}$ be a Wiener process and $\xi(t) = L\{W(u),\ 0 \leq u \leq t\}$ be a linear nonanticipative transformation of $\{W(t)\}$ i.e. the process

$$\xi(t) = \int_0^t g(t,u)\,dW(u),\quad t \geq 0 ,\quad g \in L_2(dt).$$

In |8| it is noted that there exists L such that the process has any spectral type (1) for $t \geq \varepsilon$ where $\varepsilon > 0$ is arbitrary but fixed. The idea of the proof of this statement in |4| is to in space $H_\varepsilon^{(1)}(W)$ the mutually orthogonal wide-sense martingals $\{\zeta_n(t),\ t \geq t_o\}$, $n = \overline{1,N}$, such that (2) holds. We do not see the way to remove the restriction $t \geq \varepsilon > 0$.

Now let $H(W)$ $(H_t(W))$ be Hilbert space of all random variables η, $E\eta^2 < +\infty$, $E\eta = 0$, measurable with respect to the σ-algebra $F(F_t)$ generated by $\{W(u);\ u \geq 0\}$ $(\{W(u),\ 0 \leq u \leq t\})$. It is well-known that, $\{W(t)\}$ being a Gaussian process:

1. the conditional expectation $E(.|F_t)$ is the projection operator E_t onto $H_t(W)$,

2. the subspace $H_t^{(1)}(W)$, of $H_t(W)$ reduces $\{E_t\}$ to $\{P_t\}$. In this way, $\{E_t,\ t \geq 0\}$ is the resolution of the identity and it has its spectral type. The "part" $\{P_t\}$ of $\{E_t\}$ in $H^{(1)}(W)$

has the spectral type R_1 to which all measures equivalent to the measure $d\,\|W(t)\|^2 = dt$ belong.

In this paper we shall show that it is possible to remove the restriction $t \geq \varepsilon > 0$ in |8| if we extend the space $H^{(1)}(W)$ to $H(W)$. We find the justification for this extension in the fact that $H^{(1)}(W)$ reduces $\{E_t\}$ to $\{P_t\}$.

THEOREM 1. *There exists a continuous second order process* $\{\eta(t), t \geq 0\}$ *in* $H(W)$ *such that:*

1. $\eta(t)$ *is the nonanticipative transformation of* $\{W(u),\ 0 \leq u \leq t\}$ *i.e.* $\eta(t) = T\{W(u),\ 0 \leq u \leq t\} \in H_t(W)$ *and*

2. the spectral type of $\{\eta(t)\}$ *is any given spectral type (1).*

P r o o f. For the sake of simplicity, it is not the restriction if we suppose that the spectral types $R_n(t), n = \overline{1,N}$ are equivalent to an ordinary Lebesgue measure dt.

The existence of the process $\{\eta(t)\}$ essentially depends on the existence of the mutually orthogonal martingals $\{\zeta_n(t), t \geq 0\}$, $n = \overline{1,N}$ for which $\zeta_n(t) \in H_t(W)$ and $dF_{\zeta_n}(t) = f_n(t)dt$, $f_n(t) > 0$, a.e. . For the construction of the martingals $\{\zeta_n(t)\}$, $n = \overline{1,N}$, we shall use the Hermite polynomials of the Gaussian process. The explicit formulae for these polynominals and some of their properties are given in |5|. We consider the Hermite polynomial of degree n of the random variable $W(t)$:

$$(3)\quad H_n(W(t)) = W^n(t) + \sum_{k=1}^{[\frac{n}{2}]} (-1)^k (2k-1)!!\binom{n}{2k} t^k W^{n-2k}(t)$$

It is proved in |5| that $E_s H_n(W(t)) = H_n(W(s))$, $s < t$. It follows from this relation that $\{H_n(W(t)), t \geq 0\}$ is the martingal and that the subspace $H^{(1)}(H_n)$ reduces $\{E_s\}$. As $H_m(W(t))$ and $H_n(W(s))$ are orthogonal for $m \neq n$ we conclude that the space $H = \sum_{n=1}^{N} \oplus H^{(1)}(H_n)$ is the subspace of $H(W)$ and that

$$(4)\qquad E_s H = \sum_{n=1}^{N} \oplus H_s^{(1)}(H_n),\ s \geq 0 .$$

Now we can find, following the construction in |1|, the continuous process $\{\eta(t),\ t \geq 0\}$ for which $H_s^{(1)}(\eta) = E_s H$, $s \geq 0$. There remains to show that the spectral type of $\{\eta(t)\}$ is the given type (1). From

$$H_s^{(1)}(\eta) = \sum_{n=1}^{N} \oplus H_s^{(1)}(H_n),\ s \geq 0,$$

it follows that the spectral multiplicity of $\{\eta(t)\}$ is N. We have $F_{H_n}(t) = EH_n^2(W(t)) = n!t^n$, $t \geq 0$. It means that the spectral type of $\{H_n(W(t)),\ t \geq 0\}$, is equivalent to dt.

COROLLARY 1. *There exosts a second order continuous N-ple Markov process (in the sense of |2|)* $\{\eta(t),\ t \geq 0\}$ *such that:*

1. $\eta(t)$ *is the random variable measurable with respect to the σ-algebra generated by* $W(t)$ *i.e.* $\eta(t) = T_t(W(t))$ *for each* $t \geq 0$ *and*

2. $\{\eta(t)\}$ *has the finite spectral multiplicity* N.

We may take

$$\eta(t) = H_1(W(t)) \oplus \phi(t)H_2(W(t)) \oplus \ldots \oplus \phi^{N-1}(t)H_N(W(t))$$

where ϕ is a continuous function not absolutely continuous in any interval (|3|,|7|,|6|).

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REZIME

O SPEKTRALNOM TIPU NELINEARNE I NEANTICIPIRAJUĆE TRANSFORMACIJE VINEROVOG PROCESA

Neka je $H(W)$ Hilbertov prostor kvadrat integrabilnih funkcionela Vinerovog procesa $W(t)$, $t \geq 0$. U radu se pokazuje da postoji proces

$$\eta(t) = T\{W(u),\ 0 \leq u \leq t\},\ t \geq 0$$

koji ima unapred zadani proizvoljni spektralni tip.