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SOME DIFFERENCE SCHEMES FOR TWO POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT

This paper is concerned with discretizing the boundary value problems in ordinary differential equations. We set up a total of 4 schemes for a boundary value problem $-x''=f(t,x)$ on $[0,1]$, $R_i x=\gamma_i$ $(i=1,2,)$, with four classes of linear functionals R_i on $C^1[0,1]$ on a nonuniform mesh.

1. INTRODUCTION

We shall consider a boundary value problem of the from

(1) $-x''=f(t,x)$, $t\in I=[0,1]$, $R_i x=\gamma_i$, $i=0,1$,

with four classes of linear functionals R_i on $C^1(I)$. We assume that $f\in C(I\times\mathbb{R})$ and $\gamma_i\in\mathbb{R}$, $i=0,1$. Further assumptions will come into the discussion later.

Let $n\in\mathbb{N}$, $n\geq 3$. With $k_j\in\mathbb{R}$, $k_j>0$, $j=1,2,\ldots,n$ we define a nonuniform mesh

(2) $I_h=\{t_o=0,\ t_j=t_{j-1}+hk_j;\ j=1,2,\ldots,n\}$,

where

$$h^{-1}=\sum_{j=1}^{n}k_j .$$

For the numerical solution of problem (1) we form a discrete analogue to (1) with canonical form

(3) $A_h x=B_h F_h x+r_h[\gamma_o,\gamma_1]$ in $\mathbb{R}^{I_h}$,

where $A_h, B_h \in L(\mathbb{R}^{I_h})$ (= set of all linear operators on $\mathbb{R}^{I_h}$) and where $r_h \in L(\mathbb{R}^2, \mathbb{R}^{I_h})$. For any of our schemes F_h is the non-linear mapping to $\mathbb{R}^{I_h}$ into itself which assigns to $x \in \mathbb{R}^{I_h}$ the element $F_h x \in \mathbb{R}^{I_h}$ whose t-th component is given via

$$(F_h x)(t) = f(t, x(t)) , \quad t \in I_h .$$

Any sequence of discrete problems (3) defines a (finite difference) scheme for the boundary value problem (1).

The i-th equation of (3) reads

$$\sum_{j=0}^{n} A_h(i,j)x_j = \sum_{j=0}^{n} B_h(i,j)F_h x_j + r_h[\gamma_0,\gamma_1](i) .$$

We abbreviate this as

$$(A_h(i,0),\ldots,\underline{A_h(i,i)},\ldots,A_h(i,n)) = (B_h(i,0),\ldots,\underline{B_h(i,i)},\ldots,B_h(i,n))+r_h[\gamma_0,\gamma_1](i),$$

where we shall leave out zero entries and where we shall write the common factors of the entries of the respective matrices in front of the parentheses, see |2|.

We form the finite difference schemes for (1) by using the second order approximation of $-x''$ and $R_i x$ from |4|,|5|. Now we shall describe this formula.

Let

$$x \in C^4(I),\ h \in \mathbb{R},\ h > 0,\ \alpha_j \in \mathbb{R}\setminus\{0\}, j=1,2,3,\ \alpha_i \neq \alpha_j$$

if $i \neq j$, $i,j = 1,2,3$, and t, $t+h\alpha_j \in I$, $j=1,2,3$. Then

$$(4) \quad -x''(t) = h^{-2}(ax(t+\alpha_1 h) + bx(t) + cx(t+\alpha_2 h) + dx(t+\alpha_3 h)) + O(h^2)$$

$$(5) \quad x'(t) = h^{-1}(\hat{a}x(t+\alpha_1 h) + \hat{b}x(t) + \hat{c}x(t+\alpha_2 h)) + O(h^2)$$

where

$$a = \frac{2(\alpha_2 + \alpha_3)}{\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}, \qquad b = \frac{-2(\alpha_1+\alpha_2+\alpha_3)}{\alpha_1\ \alpha_2\ \alpha_3},$$

$$(6) \quad c = \frac{2(\alpha_1 + \alpha_3)}{\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}, \qquad d = \frac{2(\alpha_1 + \alpha_2)}{\alpha_3(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)},$$

$$(7) \qquad \hat{a} = \frac{-\alpha_2}{\alpha_1(\alpha_1-\alpha_2)} \qquad \hat{b} = -\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}, \qquad \hat{c} = \frac{-\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}$$

All the schemes described in this paper in the case that I_h is a uniform mesh with the step width $h = n^{-1}$, $n \in \mathbb{N}$, are the same as in |2|. Also properties of our matrices A_h, we shall prove by using the methods from |1|,|2|,|8|,|9|.

In |6|,|7| is also considered the fourth order Hermitian approximation of (1) and the second order approximation of

$$(8) \qquad -x'' + p(t)x = f(x), \qquad t \in I,\ R_i x = \gamma_i,\ i=0,1,\ ,\ p \in C(I) .$$

Also in |3| are given some difference schemes on a nonuniform mesh (2) for (1) and (8). So, we have 12 different schemes on a nonuniform mesh which, in the special case when the mesh is uniform, are the same as in |2|.

2. FINITE DIFFERENCE SCHEMES

In this section we describe the schemes which we are going to be discussed. We separate four cases of boundary constraints (as in |2|):

I Dirichlet conditions

$$Rx = x(0) \ , \quad R_1 x = x(1)$$

II The uncoupled boundary conditions involving derivatives at both endpoints

$$R_o x = g_o\, x(0) - x'(0), \quad R_1 x = g_1 x(1) + x'(1) \ ,$$

$$g_o \geq 0, \quad g_1 \geq 0, \quad g_o + g_1 > 0 \ .$$

III The uncoupled boundary conditions involving a derivative at one endpoint

$$R_o x = x(0), \quad R_1 x = g_1 x(1) + x'(1), \quad g_1 \geq 0 \ ,$$

$$R_o x = g_o x(0) - x'(0), \quad R_1 x = x(1), \quad g_o \geq 0,$$

respectively.

IV Sturm-Liouville conditions of the form

$$R_o x = gx(0) - g_o x'(0) + g_1 x'(1), \quad R_1 x = x(0) - x(1),$$

$$g > 0, \quad g_0 > 0, \quad g_1 > 0 .$$

Thus we assume $\gamma_1 = 0$ in (1).

In the following cases I_h, h are as in the introduction.

CASE I. Let

$$1 \le k_i \le k_{i+1}, \quad i=1,2,\dots,n-2,$$

(9)

$$k_n = \sum_{j=o}^{p_{n-1}} k_{n-1-j} \quad \text{for some } p_{n-1} \in \{0,1,\dots,n-2\} .$$

We define (with $k_{n+1} := k_n$)

(10) $\alpha_2(i) = k_{i+1}$, $\alpha_3(i) = \alpha_2(i) + k_{i+2}$, $i=1,2,\dots,n-1$,

(11) $\alpha_1(i) = -\sum_{j=o}^{p_i} k_{i-j}$, $i = 1,\dots,n-1$.

Here p_{n-1} is determined in (9) and $p_i \in \{0,1,\dots,i-1\}$, $= 1,2,\dots,n-2$ we determine so that

(12) $\alpha_1(i) + \alpha_3(i) \ge 0$, $i=1,2,\dots,n-2$.

It is shown in |4| how one can determine p_i. The choice $p_i = 0$, $i=1,2,\dots,n-1$, and $k_{n-1} = k_n$ also possible.

From (9) - (12) follows

$$\alpha_1(i) < 0, \quad 0 < \alpha_2(i) \le 0.5\alpha_3(i),$$

(13)

$$\alpha_1(i) + \alpha_3(i) \ge 0, \quad i=1,2,\dots,n-1,$$

(14) $\alpha_1(n-1) + \alpha_2(n-1) = 0$,

$$t_i + \alpha_j(i)h \in I_h, \ j=1,2; \quad i=1,2,\dots,n-1,$$

$$t_i + \alpha_3(i)h \in I_{h-1}, \ i=1,2,\dots,n-2.$$

Let a_i, b_i, c_i, d_i be determined by (6), as a,b,c,d respectively, with $\alpha_1(i)$, $\alpha_2(i)$, $\alpha_3(i)$. Then we have the following scheme.

Scheme I.

$$(\underline{1}) = (\underline{0}) + \gamma_i \quad \text{for} \quad t = 0,1,$$

$$h^{-2}(a_i, \underbrace{0,\dots,0}_{p_i}, \underline{b_i}, c_i, d_i) = (\underline{1}) \quad \text{for} \quad t_i \in I_h \setminus \{0,1,t_{n-1}\},$$

$$h^{-2}(a_{n-1}, \underbrace{0,\dots,0}_{p_{n-1}}, \underline{b_{n-1}}, c_{n-1}) = (\underline{1}) \text{for} \quad t = t_{n-1}.$$

REMARK. The assumption (9) is natural in the case when the solution x of (1) has a boundary layer property at $t=0$. If x has this property at $t=1$ we can use the scheme

$$(\underline{1}) = (\underline{0}) + \gamma_i \quad \text{for} \quad t = 0,1$$

$$h^{-2}(c_1, \underline{b_1}, \underbrace{0,\dots,0}_{p_i}, a_1) = (\underline{1}) \quad \text{for} \quad t = t_1$$

$$h^{-2}(d_i, c_i, \underline{b_i}, \underbrace{0,\dots,0}_{p_i}, a_i) = (\underline{1}) \quad \text{for} \quad t \in I_h \setminus \{0,1,t_1\},$$

with $k_1 = k_2$, $k_i \geq k_{i+1} \geq 1$, $i=2,\dots,n-1$, and $\alpha_1(i) = \alpha_2(i)$, $\alpha_3(i)$, as above.

If the solution of x has boundary layer properties at $t=0$ and $t=1$ we define

$$k_{n+i} = k_{n+1-i}, \ i=1,2,\dots,n, \quad h^{-1} = 2\sum_{j=1}^{n} k_j,$$

$$t_o=0, \ t_{i+1}=t_i + hk_{i+1}, \ i=0,1,\dots,2n-1$$

$$I_h^1 = \{t_i : i=0,1,\dots,n\}, \ I_h^2 = \{t_i : i=n+1,n+2,\dots,2n\},$$

$$-\alpha_1(n) = \alpha_2(n) = k_n, \quad \alpha_3(n) = k_n + k_{n-1}, \quad p_n = 0.$$

Now we use the scheme

$$(\underline{1}) = (\underline{0}) + \gamma_i \quad \text{for} \quad t = 0,1$$

$$h^{-2}(a_i, \underbrace{0,\dots,0}_{p_i}, b_i, c_i, d_i) = (\underline{1}) \quad \text{for} \quad t_i \in I_h^1 \setminus \{0\},$$

$$h^{-2}(d_{2n-i}, c_{2n-i}, \underline{b_{2n-i}}, \underbrace{0,\dots,0}_{p_i}, a_{2n-i}) = (\underline{1}) \quad \text{for} \quad t_i \in I_h^2 \setminus \{1\},$$

with the coefficients a_i, b_i, c_i, d_i as above.

CASE II. In this case all the assumptions are the same as in case I.

Scheme II. For $t \in I_h \setminus \{0,1\}$ as in case I, and

$$h^{-1}(-\hat{b}_o+hg_o, -\hat{a}_o, -\hat{c}_o) = (\underline{0}) + \gamma_o \quad \text{for} \quad t = 0,$$

$$h^{-1}(\hat{c}_n, 0, \dots, 0, \hat{a}_n, \hat{b}_n + hg_1) = (\underline{0}) + \gamma_1 \quad \text{for} \quad t = 1,$$

where $\hat{a}_o$, $\hat{b}_o$, $\hat{c}_o$, are given by (7) with

$\alpha_1 = -\alpha_1(1) = k_1$, $\alpha_2 = \alpha_2(1) - \alpha_1(1) = k_1 + k_2$, and

$\hat{a}_n, \hat{b}_n, \hat{c}_n$ with $\alpha_1 = -\alpha_2(n-1) = -k_n$, $\alpha_2 = -\alpha_2(n-1) +$
$+ \alpha_1(n-1) = -2k_n$. So we have

$$(15) \qquad \hat{a}_o = \frac{k_1+k_2}{k_1k_2}, \quad \hat{b}_o = -\frac{2k_1+k_2}{k_1(k_1+k_2)}, \quad \hat{c}_o = \frac{-k_1}{k_2(k_1+k_2)}$$

$$(16) \qquad \hat{a}_n = \frac{-2}{k_n}, \quad \hat{b}_n = \frac{3}{2k_n}, \quad \hat{c}_n = \frac{1}{2k_n}$$

CASE III.

Scheme III

$R_o x = x(0)$: for $t=0$ as in I and for $t \in I_h \setminus \{0\}$ as in II

$R_1 x = x(1)$: for $t=1$ as in I and for $t \in I_h \setminus \{1\}$ as in II.

CASE IV. Let $n > 3$, $I_h = \{t_o=0,\ t_j = t_{j-1}+k_j h \quad j=1,2,\dots,n-1\}$. Let k_i, $i=1,2,\dots,n$, be satisfied (9) by $p_{n-1}=0$. The other assumptions are as above.

Scheme IV. For $t \in I_h \setminus \{0, t_{n-1}, t_{n-2}\}$ as in I,

$$h^{-1}(\underline{-\hat{b}_o g_o + \hat{b}_n g_1 + gh}, \ -\hat{a}_o g_o, \ -\hat{c}_o g_o, 0, \dots, 0, \hat{c}_n g_1, \hat{a}_n g_1) = (\underline{0}) + \gamma_o \quad \text{for } t=0,$$

$$h^{-2}(d_{n-2}, 0, \dots, a_{n-2}, \underbrace{0, \dots, 0}_{p_{n-2}}, \underline{b_{n-2}}, c_{n-2}) = (\underline{1}) \quad \text{for } t=t_{n-1} \text{ and } p_{n-2} < n-3,$$

$$h^{-2}(d_{n-2}+a_{n-2}, 0, \dots, 0, \underline{b_{n-2}}, c_{n-2}) = (\underline{1}) \quad \text{for } t=t_{n-2} \text{ and } p_{n-2} = n-3,$$

$$h^{-2}(c_{n-1},0,\ldots,0,a_{n-1},\underline{b_{n-1}}) = (\underline{1}) \quad \text{for } t=t_{n-1}\,.$$

3. PROPERTIES OF THE SCHEMES I - IV

We begin with some properties of the coefficients of schemes I-IV.

From (6) and (13) it follows that

$$a_i<0,\quad b_i>0,\quad c_i\le 0\,,$$

(17)
$$d\begin{cases} \le 0 & \text{for } \alpha_1(i)+\alpha_2(i)\le 0,\\ =0 & \text{for } \alpha_1(i)+\alpha_2(i)>0\,, \end{cases}$$

and from (7)

$$(0<\alpha_1(i)<\alpha_2(i)) \Longrightarrow (\hat a_i>0,\ \hat b_i<0,\ \hat c_i<0),$$

(18)
$$(\alpha_1(i)<0,\ \alpha_1(i)+\alpha_2(i)\ge 0) \Longrightarrow \hat a_i<0,\ \hat b_i\ge 0,\ \hat c_i>0)\,,$$

$$(\alpha_1(i)<\alpha_2(i)<0) \Longrightarrow (\hat a_i>0,\ \hat b_i>0,\ \hat c_i<0)\,.$$

In case $\alpha_1(i)+\alpha_2(i)=0$, we have

(19)
$$\hat b_i=0,\quad \hat a_i=-\hat c_i=-(2\alpha_2(i))^{-1}$$
$$d_i=0,\quad a_i=c_i=-0.5b_i=-\alpha_2(i)^{-2}\,,$$

and for $\alpha_1(i)+\alpha_3(i)=0$

(20)
$$c_i=0,\quad a_i=d_i=-0.5b_i=-\alpha_3(i)^{-2}$$

All our schemes are based on scheme I, and it serves as a basis for our study. The matrix A_h from scheme I is

$$A_h = h^{-2}\begin{bmatrix} h^2 & & & & & \\ a_{10} & b_1 & c_1 & d_1 & & \\ a_{21} & a_{20} & b_2 & c_2 & d_2 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{n-2,n-3} & a_{n-2,n-4}\cdots & a_{n-2,0} & b_{n-2} & c_{n-2} & d_{n-2} \\ a_{n-1,n-2} & a_{n-1,n-3}\cdots & & a_{n-1,0} & b_{n-1} & c_{n-1} \\ & & & & & h^2 \end{bmatrix}$$

where for $i=1,2,\dots,n-1$, $j=0,1,2,\dots,i-1$,

$$a_{ij} = \begin{cases} a_i & \text{for} \quad j = p_i \\ 0 & \text{for} \quad j \neq p_i \end{cases}$$

Only the coefficients $A_h(i,i+2) = d_i$, $i=1,2,\dots,n-2$ can be positive. If we define

$$\tau_d^+ = \{i:\ \alpha_1(i) + \alpha_2(i) > 0,\quad i=1,2,\dots,n-2\}$$

we have

$$d_i > 0 \quad \text{if} \quad i \in \tau_d^+ .$$

Now from (17) if follows (with $d_{n-1} = 0$) that

$$a_i < 0,\ c_i \begin{cases} < 0 & \text{if} \quad d_i \geq 0 \ , \\ \leq & \text{if} \quad d_i > 0 \ . \end{cases}$$

We are using notations from |1|,|2| (see also |3|) and define $A_{h,i}$, $i=1,2,3,4$, as a matrix from scheme I, II, III or IV.

THEOREM 1. *The matrix* $A_{h,i}$, $i=1,2,3,4$ *is inverse monotone if* $\tau_d^+ \neq \emptyset$ *and if for each* $i \in \tau_d^+$

(21) $\qquad \alpha_1(i+1) \geq z_1$

where z_1 *is smaller solution of the equation*

$$z^2 + \alpha_3(i+1)z + \alpha_2(i+1)\Delta = 0$$

with

$$\Delta^{-1} = \frac{-1}{\alpha_3(i+1)+\alpha_2(i+1)} + \frac{\alpha_3(i+1)\alpha_3(i)(\alpha_3^2(i) - \alpha_1^2(i))}{4\ \alpha_2(i)(\alpha_2^2(i)-\alpha_1^2(i))(\alpha_3^2(i+1)-\alpha_2^2(i+1))}$$

If $\tau_d^+ = \emptyset$ *then* $A_{h,1}$ *is an M-matrix.*

P r o o f. In case $A_{h,1}$ the proof is given in |5|. Now we are going to prove that $A := h^2 A_{h,2}$ is inverse monotone i.e. that $A_{h,2}$ is inverse monotone. In other cases the proof is analogous. We are using the notations and theorems from |1| |2|,|3|. Let $A_d = \mathrm{diag}(h^2, b_1, \dots, b_{n-1}, h^2)$,

$$\tilde{d}_i = \begin{cases} d_i & \text{for } i \in \tau_d^+ \\ 0 & \text{for } i \notin \tau_d^- \end{cases} \quad i=1,2,\dots,n-2$$

$$A_o^+ = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -\hat{c}_o h \\ & 0 & 0 & \tilde{d}_1 & & \\ & & 0 & 0 & \tilde{d}_2 & \\ & & & \ddots & \ddots & \ddots \\ & & & 0 & 0 & \tilde{d}_{n-2} \\ & & & 0 & 0 & 0 \\ \hat{c}_{n,n-1} & \hat{c}_{n,n-2} & \dots & \hat{c}_{n0} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -\hat{a}_o h & & & & & \\ 0.5a_1 & 0 & g_1 c_1 & d_1 - \tilde{d}_1 & & & \\ a_{21} & a_{20} & 0 & 0.5c_2 & d_2 - \tilde{d}_2 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ a_{n-2,n-3} & & & a_{n-2,0} & 0 & 0.5c_{n-2} & d_{n-2} - \tilde{d}_{n-2} \\ qa_{n-1,n-2} & & & qa_{n-1,1} & qa_{n-1,0} & 0 & 0.5c_{n-1} \\ 0 & & & 0 & 0 & \hat{a}_n h & 0 \end{bmatrix}$$

where for some $q \in (0, 0.5)$

$$g_1 = \begin{cases} 0.5 & \text{for } k_1 < k_2, \\ q & \text{for } k_1 = k_2 \end{cases}, \quad a_{ij} = \begin{cases} 0.5a_i & \text{for } j = p_i \\ 0 & \text{for } j \neq p_i \end{cases} \quad \begin{matrix} i=1,2,\dots,n-1, \\ j=0,1,\dots,i-1, \end{matrix}$$

$$\hat{c}_{nj} = \begin{cases} 0 & \text{for } j \neq p_{n-1} \\ \hat{c}_n h & \text{for } j = p_{n-1} \end{cases} \quad j=0,1,\dots,n-1$$

Let $A^- = A - A_d - A_o^+$, $C = A^- - B$, then $B \le 0$ and $C \le 0$. The ML-criterion yields that A is inverse monotone if

(i) $A \le ML$, where

$$M = A_d + B \ , \quad L = E + A_d^{-1} C \, ,$$

(ii) M is an M-matrix and $L_o \le 0$,

(iii) there exists $e > 0$ such that $Ae \ge 0$ and M or L connects $\tau^o(Ae)$ with $\tau^+(Ae)$.

The estimate $A \le ML$ holds if and only if

$$(22) \qquad \begin{aligned} &4d_i b_{i+1} \le c_i c_{i+1} \, , \quad \text{for} \quad i \in \tau_d^+ \, , \\ &-\hat{c}_o h b_1 \le -(1-q_1)\hat{a}_o h c_1 \, , \\ &\hat{c}_n h b_{n-1} \le (1-q) a_{n-1} \hat{a}_n h \end{aligned}$$

The estimate (22) is proved in |3|, and the following two can be easily proved.

With $\delta = (1,1,\dots,1,1)$ we have

$$\tau^o(M\delta) = \begin{cases} \{0\} & \text{if} \quad g_o = 0, \quad g_1 > 0 \, , \\ \emptyset & \text{if} \quad g_o > 0, \quad g_1 > 0 \, , \\ \{n\} & \text{if} \quad g_o > 0, \quad g_1 = 0 . \end{cases}$$

Since $M_o = B \le 0$ and M connects $\tau^o(M\delta)$ with $\tau^+(M\delta) = \{0,1,\dots,n\} \setminus \tau^o(M\delta)$ from the M-criterion it follows that M is an M-matrix. Now we see that (ii) is satisfied since $L_o = A_d^{-1} C \le 0$.

In case II we have $g_o \ge 0$, $g_1 \ge 0$, $g_o + g_1 > 0$, and

$$\tau^+(A\delta) = \begin{cases} \{0\} & \text{if} \quad g_o > 0, \quad g_1 = 0, \\ \{0,n\} & \text{if} \quad g_o > 0, \quad g_1 > 0 \, , \\ \{n\} & \text{if} \quad g_o = 0, \quad g_1 > 0 \, . \end{cases}$$

In first two cases the matrix M connects the set $\tau^o(A\delta) = \{0,1,\dots,n\} \setminus \tau^+(A\delta)$ with $\tau^+(A\delta)$ so that for $i_o \in \tau^o(A\delta)$ we define

$$\begin{aligned} &i_{j+1} = i_j - 1 - p_{i_j}, \quad j=0,1,\dots,r-1, \\ &i_r = 0, \end{aligned}$$

where $i_o = n \Rightarrow p_{i_o} = 0$.

In the case that $g_o = 0$, $g_1 > 0$ the matrix M connects $\tau^o(A\delta) = \{0,1,\dots,n-1\}$ with $\tau^+(A\delta) = \{n\}$ so that for $i_o \in \tau^o(A\delta)$ we form

$$i_{j+1} = i_j + 1, \quad j=0,1,\dots,r-1$$
$$i_r = n.$$

Now from the ML-criterion it follows that A is an inverse monotone matrix.

So, we have $A_{h,i}^{-1} \geq 0$, $i=1,2,3,4$ and $B_h = \mathrm{diag}(0,1,\dots \dots,1,0)$ in each case I-IV. Before giving Theorem 2 we shall introduce some notations. Let

$$Q_1 = b_1 - c_1(1-z)(1+k_2/k_1)^2 ,$$
$$z \in (0,0.5) \quad \text{if} \quad k_1 = k_2 ,$$
$$z = 0.5 \quad \text{if} \quad k_1 < k_2 ,$$

$$Q_i = \begin{cases} \infty & \text{if} \quad i-1 \notin \tau_d^+ \\ \dfrac{c_i c_{i-1}}{4d_{i-1}} - b_i, & \text{if} \quad i-1 \in \tau_d^+ \end{cases} \qquad i=2,3,\dots,n-1 ,$$

$$S = b_{n-1}(1-2z).$$

THEOREM 2. *If for* $i \in \tau d^+$ (21) *is satisfied then there exists the smallest positive eigenvalue* $\lambda_{h,j}$ *for the eigenvalue problem* $A_{h,j}x = \lambda B_h x$, $j=1,2,3,4$, *and the matrices* $A_{h,j} - B_h D_h$, $j=1,2,3,4$ *are inverse-monotone for any diagonal matrix* $D_h = \mathrm{diag}(\mu_o,\mu_1,\dots,\mu_n)$ *whose diagonal elements are all in* $[-\bar{q}_i h^{-2}, \lambda_{h,j})$, $i=1,2,\dots,n-1$. *The following table shows a* $\bar{q}_i$ *of this type where* $\bar{q} = \infty$ *means that* $[-h^{-2}\bar{q}, \lambda_{h,j}) = (-\infty, \lambda_{h,j})$.

Scheme	I,III ($R_o x = x(0)$)	II,III, ($R_1 x = x(1)$), IV
$\bar{q}_1$	∞	Q_1
$\bar{q}_i$ ($i=2,3,\dots,n-2$)	Q_i	Q_i
$\bar{q}_{n-1}$	Q_{n-1}	$\min(S, Q_{n-1})$

The proof of this Theorem is based on the inverse monotonicity of matrices $A_{h,j}$ and it is analogous to the proof of PO from |2|.

REMARK. If we take in (11) $p_i=0$, $i=1,2,\ldots,n-1$ and

$$k_j = \begin{cases} 1 & \text{for } j=1,2,\ldots,p-1 \\ k & \text{for } j=p,\ldots,n \end{cases} \qquad 1<p<n,$$

then we have

$$\bar{q}_i = \begin{cases} \infty & \text{if } i \neq p, \\ \dfrac{3}{2k^2(k^2-1)} & \text{if } i = p. \end{cases}$$

Now, we shall consider the nonlinear system (3) with the assumption:

(23) $$q(v-w) \le f(t,v) - f(t,w) \le \mu(v-w), \quad v,w \in \mathbb{R},\ w \le v$$

for some $q,\mu \in \mathbb{R}$, where

(24) $$-h^{-2}\bar{q} \le \mu \le \lambda_h$$

λ_h is the smallest eigenvalue to $A_h x = \lambda B_h x$,

$$\bar{q} = \min\{\bar{q}_i : i=1,2,\ldots,n-1\}.$$

Finally, we shall let $h>0$ be so small that

(25) $$-h^{-2}\bar{q} < \rho_h := 0.5(\lambda_h + q).$$

Let the mapping $T_h = A_h - B_h F_h$ be defined by the schemes of cases I-IV.

THEOREM 3. *Let for* $i \in \tau_d^+$ (21) *be satisfied and let* (23), (24), (25) *be satisfied. Then* T_h^{-1} *is* $(A_h - \mu B_h)^{-1}$ *- bounded and for any diagonal matrix* $D_h = \mathrm{diag}(d_o, d_1, \ldots, d_n)$ *with* $d_i \in [-h^{-2}\bar{q}_i, \rho_h)$, $i=1,2,\ldots,n-1$, *the parallel chord method*

$$x^o \in \mathbb{R}^{I_h}, \quad (A_h - B_h D_h)x^{k+1} = B_h(F_h - D_h)x^k + r_h; \quad k \in \mathbb{N}$$

converges for any initial approximation to the only solution of the system $Tx = r_h$.

We have, using the second order approximation of $-x''$: if $1 \le k_i \le k_{i+1} \le k_o$, $i=1,2,\dots,n-1$, for some fixed $k_o \in \mathbb{R}$, then $h \to 0$ if $n \to \infty$. Now we shall prove that we have the implication

$$(26) \qquad x \in C^4(I) \implies \|x_h - x^h\|_\infty = O(h^2),$$

where x is the solution of the continuous problem (1), the vector $x_h \in \mathbb{R}^{I_h}$ stands for the restriction of x to the mech I_h, and x^h is the solution of (3) in each case I-IV.

THEOREM 4. *Let the assumptions of Theorem 3 be satisfied, then (26) is true for each case I-V.*

P r o o f. Let $\bar\lambda$ be the smallest positive eigenvalue of the eigenvalue problem

$$-x'' = \lambda x \quad \text{an } I, \quad R_i x = 0, \quad i=0,1 .$$

The discrete eigenvalues λ_h tend to $\bar\lambda$ as $h \to 0$ (i.e. as $n \to \infty$). So from (24) it follows that $\mu < \bar\lambda$, for h sufficiently small.

Using the technique from |2| we obtain that $\|(A_h - \mu B_h)^{-1}\|_\infty$ is uniformly bounded, i.e. there exists $\sigma \ge 0$ independent of h such that $\|(A_h - B_h)^{-1}\|_\infty \le \sigma$ for h sufficiently small. Hence we have

$$\|x-y\|_\infty \le \sigma \|T_h x - T_h y\|_\infty \quad \text{for} \quad x,y \in \mathbb{R}^{I_h}$$

if h is small enongh.

Now with $x = x_h$, $y = x^h$ we have

$$|x_h - x^h| \le (A_h - \mu B_h)^{-1} |T_h x_h - r_h| ,$$

i.e.

$$\|x_h - x^h\|_\infty \le \text{const} \cdot h^2 .$$

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REZIME

NEKE DIFERENCNE ŠEME ZA KONTURNE PROBLEME

U radu se daju 4 diferencne šeme na neekvidistantnoj mreži (2) za numeričko rešavanje problema (1). U ekvidistantnom slučaju one se svode na dobro poznate šeme, videti na primer |2|. Za matrice A_h diskretnih analogona (3) formiranih pomoću ovih šema dokazano je da su inverzno monotone. Na osnovu toga kao i u |2| za ekvidistantan slučaj, dati su uslovi pod kojima postupak paralelne sečice (teorema 3) za rešavanje diskretnog analogona konvergira i numeričko rešenje teži kontinualnom kada broj tačaka mreže teži beskonačnosti.

U radu su korišćene oznake i teoreme iz |2|, koje se mogu naći i u |3| iz ove knjige Zbornika.