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ON A NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM USING AN OPTIMAL NUMERICAL DIFFERENTIATION

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ABSTRACT

In this paper we consider a numerical solution of two boundary value problems: (BVP1) $-u''(x)=f(x,u)$, $u(0)=\gamma_0$, $u(1)=\gamma_1$, and (BVP2) $-u''(x)-q(x)u(x)=f(x)$, $u(0)=\gamma_0$, $u(1)=\gamma_1$, using the authors' four-point rule of degree 3 for the second derivative. The discretisation meshes depend upon this rule. The discrete problem to (BVP1) has the usual form, but the discrete problem to (BVP2) has an unusual form. Under certain assumptions, our schemes have a third order of convergence.

1. INTRODUCTION

We consider two boundary value problems (BVP1) and (BVP2).

(BVP1) $\quad -u''(x)=f(x,u)$ on $[0,1]$, $u(0)=\gamma_0$, $u(1)=\gamma_1$,

where $f\in C^3([0,1]\times \mathbb{R})$ and $\gamma_0,\gamma_1\in\mathbb{R}$. The nonlinearity $f(t,v)$ is assumed to satisfy the Lipschitz condition

(L) $\quad q(v-w)\le f(x,v)-f(x,w)\le \mu(v-w)$, $v,w\in\mathbb{R}$, $v\ge w$, $x\in[0,1]$

for some reals $q,\mu\in\mathbb{R}$, where

(1) $$-h^{-2}q_0\le\mu<\lambda_0,\quad q_0=3(\sqrt{5}-1),\ 0<\lambda_0<8,$$

and we let $h=(1+n\frac{3+\sqrt{5}}{2})^{-1}$ be so small, i.e. $n\in\mathbb{N}$ so great, that

(2) $$-h^{-2}q_0<\frac{1}{2}(\lambda_0+q)$$

(BVP2) $-u''-q(x)u = f(x)$ on $[0,1]$, $u(0)=\beta_o$, $u(1)=\beta$,

where $q,f \in C^3[0,1]$, $\beta_o,\beta_1 \in \mathbb{R}$. We aasume

(3) $0 \leq q(x) \leq \mu < 8$ on $[0,1]$.

The assumptions for (BVP1) are similar to the assumptions from |1|, but here q_o and λ_o are different. The parametar h depends on the rule we used for the discretisation of (BVP1). For q(x) from (BVP2) a usual assumption is $q(x) \leq 0$ in $[0,1]$. But, our scheme for (BVP2) has a special form and we have the assumption $0 \leq q(x) < 8$.

The proofs in section 3 are based upon the results of this type in |1|,|2|,|7|,|8|.

2. FINITE DIFFERENCE SCHEMES

In this section we shall describe the schemes which we are going to discuss. It is convenient to distinguish (BVP1) and (BVP2). First we shall consider (BVP1).

For $u \in C^5[0,1]$, the authors´ four-point rule of degree 3 for the second derivative, see |4|, is

$$(4)\quad -u''(x) = h^{-2}\left[af(x\pm h) + bf(x) + cf\left(x \mp \frac{1+\sqrt{5}}{2}h\right) + df\left(x \mp \frac{3+\sqrt{5}}{2}h\right)\right] + O(h^3),$$

where

$$a = -\frac{2\sqrt{5}}{5}, \quad b = 6-2\sqrt{5}, \quad c = -(3-\sqrt{5}), \quad d = \frac{7\sqrt{5}}{5} - 3 .$$

We let for $n \in \mathbb{N}$,

$$I_h = \{x_{2i} = i\,\frac{3+\sqrt{5}}{2}h,\ x_{2i+1} = h + x_{2i},\quad i=0,1,\dots,n\},$$

where $h = (1 + n\,\frac{3+\sqrt{5}}{2})^{-1}$. Now we can form the descrete analogue to (BVP1) by using (4):

$$\text{(DBVP 1)}\qquad (L_h y)_i = \begin{cases} \gamma_o, & i=0,\\ 0, & i=1,2,\dots,2n,\\ \gamma_1, & i=2n+1, \end{cases}$$

where

$$(L_h y)_i = \begin{cases} y_i, & i=0,2n+1, \\ h^{-2}[dy_{i-2}+cy_{i-1}+by_i+ay_{i+1}]-f(x_i,y_i), & i=2m,\ m=1,\dots,n, \\ h^{-2}[ay_{i-1}+by_i+cy_{i+1}+dy_{i+2}]-f(x_i,y_i), & i=2m+1,\ m=0,1,\dots n-1. \end{cases}$$

We can write the discrete problem to (BVP1) in the canonical form

$$A_h y - F_h y = 0,$$

where

$$(5) \quad A_h = h^{-2}\begin{bmatrix} h^2 & & & & & & & & \\ a & b & c & d & & & & & \\ d & c & b & a & & & & & \\ & & a & b & c & d & & & \\ & & d & c & b & a & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & d & c & b & a & & \\ & & & & & a & b & c & d \\ & & & & & d & c & b & a \\ & & & & & & & & h^2 \end{bmatrix} \in \mathbb{R}^{2n+1,2n+2}$$

and F_h is a nonlinear mapping of R^{2n+2} into itself which assigns to $y = [y_o,y_1,\dots,y_{2n+1}]^T$ the element $F_h y = [\gamma_o,f_1,f_2,\dots$ $\dots,f_{2n},\gamma_1]^T$, with $f_i = f(x_i,y_i)$.

For the discretisation of (BVP2) we use our four-point rule of degree 3 for the second derivative, |5|,

$$(6) \quad -u''(x) = k^{-2}[Af(x\mp \frac{9-\sqrt{15}}{6}k) + Bf(x\pm\frac{\sqrt{15}-3}{6}k) +$$
$$+ Cf(x\pm\frac{3+\sqrt{15}}{6}k) + Df(x\pm\frac{9+\sqrt{15}}{6}k)] + O(k^3),\ u\in C^5[0,1],$$

where

$$A = -\frac{3+\sqrt{15}}{6},\quad B = \frac{1+\sqrt{15}}{2},\quad C = -\frac{\sqrt{15}-1}{2},\quad D = \frac{\sqrt{15}-3}{6}.$$

In this case, the mesh is

$$J_k = \{x_{2i} = ik,\ x_{2i+1} = x_{2i} + \frac{9-\sqrt{15}}{6}k,\ i=0,1,\dots,n\},$$

where $k^{-1} = n + \frac{9-\sqrt{15}}{6}$, $n\in\mathbb{N}$.

Using (6) we obtain the discrete analogue to (BVP2):

$$\text{(DBVP2)} \qquad (T_k z)_i = \begin{cases} \beta_o, & i=0, \\ f_i^k, & i=1,2,\ldots,2n, \\ \beta_1, & i=2n+1, \end{cases}$$

where

$$(T_k z)_i = \begin{cases} z_i, \quad i=0,2n+1, \\ k^{-2}\left[Az_{i-1}+Bz_{i+1}+Cz_{i+3}+Dz_{i+5}\right]-q_i z_i, \ i=2m+1, \ m=0,1,\ldots,n-3, \\ k^{-2}\left[Dz_{i-5}+Cz_{i-3}+Bz_{i-1}+Az_{i+1}\right]-q_i z_i, \ i=2m, \ m=3,4,\ldots,n, \\ k^{-2}\left[A_1 z_o+B_1 z_1+C_1 z_3\right]-q_2 z_2 \quad, \ i=2, \\ k^{-2}\left[C_1 z_{2n-2}+B_1 z_{2n}+A_1 z_{2n+1}\right]-q_{2n-1} z_{2n-1}, \ i=2n-1 \\ k^{-2}\left[A_2 z_1+B_2 z_3+C_2 z_5\right]-q_4 z_4, \ i=4, \\ k^{-2}\left[C_2 z_{2n-4}+B_2 z_{2n-2}+A_2 z_{2n}\right]-q_{2n-3} z_{2n-3}, \ i=2n-3, \end{cases}$$

and

$$A_1 = -\frac{414+2\sqrt{15}}{132}, \qquad B_1 = \frac{623-20\sqrt{15}}{132}, \qquad C_1 = -\frac{19-2\sqrt{15}}{12},$$

$$A_2 = -\frac{15\sqrt{15}-8}{12}, \qquad B_2 = -2A_2, \qquad C_2 = A_2,$$

$$f^k = [\beta_o, f_1, f_2, \ldots, f_{2n}, \beta_1]^T, \ f_i = f(x_i), \ q_i = q(x_i), \ x_i \in J_k,$$

$$z = [z_o, z_1, \ldots, z_{2n+1}]^T.$$

The discrete problem to (BVP2) we write in the form

$$\text{(7)} \qquad B_k z = f_k,$$

where

$$\text{(8)} \quad B_k = k^{-2} \begin{bmatrix} k^2 \\ A_1 & B_1 & -k^2 q_2 & C_1 \\ A_1 & -k^2 q_1 & B & 0 & C & 0 & D & & & & & & 0 \\ & A_2 & 0 & B_2 & -k^2 q_4 & C_2 \\ & & A & -k^2 q_3 & B & 0 & C & 0 & D \\ & D & 0 & C & 0 & B & -k^2 q_6 & A \\ & & & & A & -k^2 q_5 & B & 0 & C & 0 & D \\ & & & D & 0 & C & 0 & B & -k^2 q_8 & A \\ & & & & & & A & -k^2 q_{2n-5} & B & 0 & C & 0 & D \\ & 0 & & & & D & 0 & C & 0 & B & -k^2 q_{2n-2} & A \\ & & & & & & & & C_2 & -k^2 q_{2n-3} & B_2 & 0 & A_2 \\ & & & & & & & D & 0 & C & 0 & B & -k^2 q_{2n} & A \\ & & & & & & & & & & C_1 & -k^2 q_{2n-1} & B_1 & A_1 \\ & & & & & & & & & & & & & k^2 \end{bmatrix}$$

(9) $f_k = [\beta_0, f_2, f_1, f_4, f_3, \ldots, f_{2n-3}, f_{2n}, f_{2n-1}, \beta_1]^T$.

3. PROPERTIES OF THE SCHEMES

In this section we shall considere the discrete analogues (DBVP1) and (DBVP2) to (BVP1) and (BVP2). We shall prove that (DBVP1) and (DBVP2) have unique solutions, say y and z, for which we have

$$\| u^h - y \|_\infty \le M_1 h^3, \quad \| u^k - z \|_\infty \le M_2 k^3 ,$$

where u^h and u^k denote the restriction of the exact solution of (BVP1) and (BVP2) to the mesh I_h and J_k, and M_1, M_2 are the constants indepedent of h and k.

We shall begin with some notations (see |1|,|2|,|7|,|8|). For $x,y \in \mathbb{R}^m$, we write

$$x \le (<) y \quad \text{iff} \quad x_i \le (<) y_i , \quad i=1,2,\ldots,m,$$
$$|x| = [|x_1|, |x_2|, \ldots, |x_m|]^T .$$

Any $e \in \mathbb{R}^m$, $e > 0$, defines the norm $\|x\|_e = \max_{1\le i\le m} \frac{|x_i|}{e_i}$ on $\mathbb{R}^m$. In particular $e = [1,1,\ldots,1]^T$ yields the maximum norm $\| \ \|_\infty$.

A mapping F of $\mathbb{R}^m$ into itself is called monotone if $x \le y \Rightarrow Fx \le Fy$ for any $x,y \in \mathbb{R}^m$.

For mappings F,G of $\mathbb{R}^m$ into itself we write $F \le G$ iff $G - F$ is monotone.

The set of matrices of the format $m \times m$ is denoted by $\mathbb{R}^{m,m}$.

For any $A = [a_{ij}] \in \mathbb{R}^{m,m}$ the matrices A_d, A_a, $A_a^- \in \mathbb{R}^{m,m}$ are defined via $A_d = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{mm})$, $A_a = A - A_d$, $A_a^- = [a_{ij}^-]$,

$$a_{ij}^- = \begin{cases} a_{ij} & \text{if} \quad a_{ij} < 0, \\ 0 & \text{if} \quad a_{ij} \ge 0. \end{cases}$$

Let $\tau^0(x) = \{i: i=1,2,\ldots,m, \ x_i = 0\}$, $\tau^+(x) = \{i: i=1,2,\ldots,m, x_i > 0\}$ for $x \in \mathbb{R}^m$.

If τ^1 and τ^2 are disjoint subsets of $\{1,2,\ldots,m\}$ we say that $A=[a_{ij}]\in \mathbb{R}^{m,m}$ connects τ^1 with τ^2 if for all $i\in\tau^1$ there are point $i_o=i, i_1, i_2, \ldots, i_r \in \{1,2,\ldots,m\}$ such that $a_{i_{j-1}\, i_j} \neq \neq 0$, $j=1,2,\ldots,r$ and $i_r\in\tau^2$.

The matrix A is called an L-matrix if $A_a \leq 0$.

$A\in\mathbb{R}^{m,m}$ is called an inverse-monotone matrix if A has an inverse $A^{-1}\geq 0$. Throughout this paper we shall use the abbreviation i.m. for inverse-monotone. The inverse-monotone L-matrix is called a M-matrix.

The following 5 theorems and their proofs can be found in |2|,|3|,|6|,|7|.

THEOREM 1. *Let* $A\in\mathbb{R}^{m,m}$, $\delta\in\mathbb{R}^m$. *If* $A_a\leq 0$, $\delta\geq 0$, $A\delta\geq 0$ *and if* A *connects* $\tau^o(A\delta)$ *with* $\tau^+(A\delta)$ *then* A *is an* M-*matrix.*

THEOREM 2. *Let* $A=[a_{ij}]\in\mathbb{R}^{m,m}$, $A_{\bar a}=A^z+A^s$, $A^z = [a_{ij}^{(z)}]\leq 0$, $A^s=[a_{ij}^{(s)}]\leq 0$. *The matrix* A *is i.m. if the following conditions are satisfied:*

1. A_d+A^z *is an* M-*matrix,*
2. $a_{ij}\leq \sum_{k=1}^{m} a_{ik}^{(z)} a_{kk}^{-1} a_{kj}^{(s)}$ *for all* $a_{ij}>0$, $i\neq j$,
3. *there exists* $e\geq 0$ $(e\in\mathbb{R}^m)$ *such that* $Ae\geq 0$ *and* A^z *or* A^s *connects* $\tau^o(Ae)$ *with* $\tau^+(Ae)$.

THEOREM 3. *Let* $A\leq B$ $(A,B\in\mathbb{R}^{m,m})$ *and assume that* B *is i.m. Then* A *is i.m. iff there exists* $e>0$ $(e\in\mathbb{R}^m)$ *such that* $B^{-1}Ae>0$.

THEOREM 4. *Let* $C\leq A\leq B$ $(A,B,C\in\mathbb{R}^{m,m})$. *If* B *and* C *are i.m. then* A *is i.m.*

THEOREM 5. *Let* F *be a nonlinear mapping* $\mathbb{R}^m$ *into itself and let* $P,Q\in\mathbb{R}^{m,m}$ *be such that* $Q\leq P$, $Q\leq F\leq P$. *Let* $A,S\in \in\mathbb{R}^{m,m}$ *be such that* A-P, A-S *are i.m.,* $2S\leq P+Q$. *Then* $(A-F)^{-1}$ *exists and* $|(A-F)^{-1} - (A-F)^{-1}w|\leq (A-P)^{-1}|v-w|$, *for any* $v,w\in\mathbb{R}^m$. *Furthermore, the parallel chord iteration*

$$x^o\in\mathbb{R}^m,\quad (A-S)x^n = (F-S)x^{n-1},\quad n\in\mathbb{N}$$

converges for any $x^o\in\mathbb{R}^m$ *to the unique solution of* $Ax=Fx$.

Now we shall consider (DBVP1).

THEOREM 6. *The matrix* A_h *from (5) is i.m. and for any matrix* $D=\mathrm{diag}(0,\mu_1,\mu_2,\dots,\mu_{2n},0)$ *such that*

(10) $\mu_i \in [-h^{-2}q_o,\lambda_o),\ i=1,2,\dots,2n,\quad q_o=3(\sqrt{5}-1),\quad 0<\lambda_o<8,$

the matrix A_h-D *is i.m.*

P r o o f. We can write the matrix A_h in the form

$$A_h = h^{-2}ML,$$

where

$$M=\begin{bmatrix} h^2 &&&&&&&& \\ & m_1 & m_2 &&&&&& \\ & m_2 & m_1 &&&&&& \\ &&& m_1 & m_2 &&&& \\ &&& m_2 & m_1 &&&& \\ &&&&& \ddots &&& \\ &&&&&& m_1 & m_2 & \\ &&&&&& m_2 & m_1 & \\ &&&&&&&& h^2 \end{bmatrix},\quad L=\begin{bmatrix} 1 &&&&&&&& \\ \ell_2 & 1 & \ell_1 &&&&&& \\ & \ell_1 & 1 & \ell_2 &&&&& \\ && \ell_2 & 1 & \ell_1 &&&& \\ &&& \ell_1 & 1 & \ell_2 &&& \\ &&&& \ddots &&&& \\ &&&&& \ell_2 & 1 & \ell_1 & \\ &&&&&& \ell_1 & 1 & \ell_2 \\ &&&&&&&& 1 \end{bmatrix}$$

$M,L\in\mathbb{R}^{2n+2,2n+2}$, $m_1=1+\frac{\sqrt{5}}{5}$, $m_2=-2(1-\frac{2\sqrt{5}}{5})$, $\ell_1=-\frac{3-\sqrt{5}}{2}$, $\ell_2=-\frac{\sqrt{5}-1}{2}$.

Since $\delta=[1,1,\dots,1]^T>0$, $M\delta=[h^2,\sqrt{5}-1,\dots,\sqrt{5}-1,h^2]^T>0$, Theorem 1 yields that M is i.m. Taking δ again, we have $L\delta = [1,0,0,\dots,0,1]^T\geq 0$, $\tau^o(L\delta)=\{1,2,\dots,2n\}$, $\tau^+(L\delta)=\{0,2n+1\}$. The matrix L connects $\tau^o(L\delta)$ with $\tau^+(L\delta)$ (for $i_o\in\tau^o(L\delta)$ we define $i_{j+1}=i_j-1$, $j=0,1,\dots,r-1$, $i_r=0$) and Theorem 1 yields that L is i.m. The matrix A_h is the product of two i.m. matrices M and L, i.e. A_h is an i.m. matrix.

Let $e\in\mathbb{R}^{2n+2}$ be defined by

$$e_i = x_i(1-x_i)+\xi,\quad x_i\in I_h,\quad \xi=\frac{8-\lambda_o}{4\lambda_o}>0.$$

Now we have $e>0$, $(A_he)_o=(A_he)_{2n+1}=\xi$, $(A_he)_i=\lambda_ie_i$, $i=1,2,\dots,2n$, where

$$\lambda_i = \frac{2}{x_i(1-x_i)+\xi} \geq \frac{2}{0.25+\xi} = \lambda_o, \quad i=1,2,\ldots,2n.$$

Let $D_1 = \mathrm{diag}(0,d_1,d_2,\ldots,d_{2n},0) \geq 0$ satisfy condition (10). Then $A_h - D_1 \leq A_h$, $(A_h-D_1)e > 0$. Since $A_h^{-1} \geq 0$ it follows $A_h^{-1}(A_h-D_1)e > 0$ and Theorem 3 yields that A_h-D_1 is i.m. Let $D_2 = \mathrm{diag}(0,d_1',d_2',\ldots,d_{2n}',0) \leq 0$ satisfy condition (10). Let $(A_h-D_2)_a = A^z + A^s$, where $A^z = [z_{ij}]$, $A^s = [s_{ij}]$ is defined via

$$z_{ij} = h^{-2}\begin{cases} -(3-\sqrt{5}) & \text{if } i=2m,\ j=2m-1,\ m=1,2,\ldots,n, \\ & \quad\ i=2m-1,\ j=2m,\ m=1,2,\ldots,n, \\ 0 & \text{otherwise}, \end{cases}$$

$$s_{ij} = h^{-2}\begin{cases} -\frac{2\sqrt{5}}{5} & \text{if } i=2m,\ j=2m+1,\ m=1,2,\ldots,n \\ & \quad\ i=2m+1,\ j=2m,\ m=0,1,\ldots,n-1, \\ 0 & \text{otherwise}. \end{cases}$$

Then applying Theorem 1 to $\delta = [1,1,\ldots,1]^T$ we see that $(A_h-D_2)d + A^z$ is i.m. Since

$$d(b-h^2 d_i') \leq a c \iff d_i' \geq -h^{-2}q_o, \text{ and } (A_h-D_2)e \geq A_h e > 0,$$

Theorem 2 yields that A_h-D_2 is i.m.. Now let the matrix $D = \mathrm{diag}(0,\mu_1,\mu_2,\ldots,\mu_{2n},0)$ satisfy condition (10). Then there exist matrices $D_1 \geq 0$ and $D_2 \geq 0$ such that $D_2 \leq D \leq D_1$. Since $A_h - D_1 \leq A_h-D \leq A_h-D_2$, Theorem 4 yields that A_h-D is i.m.

THEOREM 7. *Suppose that conditions* (L), (1), (2) *are satisfied. Then for any* $v,w \in \mathbb{R}^{2n+2}$

$$|v-w| \leq (A_h-M)^{-1}|L_h v - L_h w|,$$

where $M = \mathrm{diag}(0,\mu,\mu,\ldots,\mu,0) \in \mathbb{R}^{2n+2,2n+2}$

THEOREM 8. *Let* $S = \mathrm{diag}(0,s_1,s_2,\ldots,s_{2n},0)$, $s_i \geq -h^{-2}q_o$, $i=1,2,\ldots,2n$; *and* $s = \max_{1\leq i\leq 2n} s_i \leq \frac{1}{2}(\mu+q)$. *Then the parallel chord iteration*

$$y^o \in \mathbb{R}^{2n+2}, \quad (A_h-S)y^m = (F_h-S)y^{m-1}, \quad m \in \mathbb{N}$$

converges for any y^o *to the unique solution of* $A_h y = F_h y$.

The proofs of Theorem 7 and Theorem 8 follow directly from Theorem 5.

THEOREM 9. *Suppose that conditions* (L),(1),(2) *are satisfied. Then*

$$\|v-w\|_e \le \frac{1}{K}\|L_hv-L_hw\|_e, \quad K=\min(\xi,\lambda_o-\mu), \text{ for any } v,w\in \mathbb{R}^{2n+2},$$

$$\|v-w\|_\infty \le \frac{2}{\lambda_o\xi K}\|L_hv-L_hw\|_\infty, \quad \text{for any } v,w\in \mathbb{R}^{2n+2}.$$

P r o o f. Since $(A_h-M)e \ge Ke$ and $(A_h-M)^{-1}\ge 0$ it follows that

$$(A_h-M)^{-1}e \le \frac{1}{K}e, \quad \|(A_h-M)^{-1}\|_e = \|(A_h-M)^{-1}e\|_e \le \frac{1}{K}.$$

From Theorem 7 it follows that

$$\|v-w\|_e \le \|(A_h-M)^{-1}\|_e\|L_hv-L_hw\|_e \le \frac{1}{K}\|L_hv-L_hw\|_e \quad \text{for any}$$

$$v,w\in \mathbb{R}^{2n+2}.$$

Since

$$\frac{1}{0.25+\xi}\|z\|_\infty \le \|z\|_e \le \frac{1}{\xi}\|z\|_\infty \quad \text{for any } z\in \mathbb{R}^{2n+2},$$

we have

$$\|v-w\|_\infty \le \frac{0.25+\xi}{\xi K}\|L_hv-L_hw\|_\infty, \quad \text{for any } v,w\in \mathbb{R}^{2n+2},$$

which completes the proof.

COROLLARY 1. *Suppose that conditions* (L),(1),(2) *are satisfied. Let* y *be the solution of* (DBVP1), u *the solution of* (BVP1) *and the vector* u^h *be the restriction of* u *to the mesh* I_h. *Then*

$$u\in C^5[0,1] \Rightarrow \|y-u^h\|_\infty \le M_1h^3$$

where M_1 *is a constant independent of* h.

P r o o f. In |5| it is proved that $u\in C^5[0,1]$ implies that

$$L_hu^h - L_hy = O(h^3).$$

Assuming $v=u^h$ and $w=y$, from Theorem 9, there, follows the proof.

We shall consider now the boundary value problem (BVP2) and its discrete analogue $B_k z = f_k$.

THEOREM 10. *Let condition* (3) *be satisfied. Then the matrix* B_k *is i.m.*

P r o o f. Let $(B_k)_a^- = B^z + B^s$, where $B^z = [b_{ij}^{(z)}]$,

$$b_{ij}^{(z)} = \begin{cases} \frac{C}{2} & \text{if } i=2m,\ j=2m+2,\ m=1,2,\dots,n-2, \\ & \quad\ i=2m+1,\ j=2m-1,\ m=2,3,\dots,n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(B_k)_d + B^z$ is an M-matrix, $2DB_2 \le A_2C$, $4BD \le C^2$ and

$$B_k e = [\xi, 2-q_2e_2, 2-q_1-e_1, 2-q_4e_4, 2-q_3e_3, \dots$$

$$\dots, 2-q_{2n}e_{2n}, 2-q_{2n-1}e_{2n-1}, \xi]^T > 0, \quad \text{for}$$

$$e = [e_o, e_1, \dots, e_{2n+1}]^T > 0,\ e_i = x_i(1-x_i) + \xi,\ x_i \in J_k,$$

where $\xi > 0$ is determined so that $\mu < \frac{2}{0.25+\xi} < 8$, we conclude applying Theorem 2, that B_k is i.m.

THEOREM 11. *Suppose that condition* (3) *is satisfied. Let* z *be the solution of* (DBVP2), u *the solution of* (BVP2) *and the vector* u^k *the restriction of* u *to the mesh* J_k. *Then*

$$u \in C^5[0,1] \implies \| u^k - z \|_\infty \le M_2 k^3,$$

where M_2 *is a constant indepedent of* k.

P r o o f. Let $\varepsilon \in (0, \min(-A_1, \frac{A_1A_2}{2B_1-A_2}))$. Now multiply the zeroth row of the matrix B_k by εk^{-2} and add the result to the first and third row. Also, multiply the (2n+1)th row of matrix B_k by εk^{-2} and add the result to the (2n)th and (2n-2)th row. Now multiply the first, third, (2n)th and (2n-2)th row of matrix B_k by k^2. As a result one obtains a new matrix $\tilde{B}_k$. The equations

$B_k z = f_k$ and $\tilde{B}_k z = \tilde{f}_k$, with

$$\tilde{f}_k = [\beta_o, k^2 f_2 + \varepsilon\beta_o, f_1, k^2 f_4 + \varepsilon\beta_o, f_3, f_6, f_5, \dots, f_{2n-2}, k^2 f_{2n-3} +$$

$$+ \varepsilon\beta_1, f_{2n}, k^2 f_{2n-1} + \varepsilon\beta_1, \beta_1]^T$$

are equivalent. Using Theorem 2, with vector e from Theorem 10, we obtain that $\tilde{B}_k$ is i.m.

Let $\lambda = \min(1, \frac{\varepsilon\xi}{0.25+\xi}, \frac{2}{0.25+\xi} - \mu) > 0$. Then $\tilde{B}_k e \geq \lambda e$ and $\| \tilde{B}_k^{-1} \|_e \leq \frac{1}{\lambda}$.

Since $\tau(u) = \tilde{B}_k u^k - \tilde{B}_k z = \tilde{B}_k u^k - \tilde{f}_k = O(k^3)$ and $u^k - z = \tilde{B}_k^{-1}\tau(u)$, we have

$$\| u^k - z \|_\infty \leq (0.25+\xi) \| u^k - z \|_e \leq (0.25+\xi) \| B_k^{-1} \|_e \| \tau(u) \|_e \leq$$

$$\leq (0.25+\xi) \frac{1}{\lambda} \cdot \frac{1}{\xi} \| \tau(u) \|_\infty \leq M_2 k^3 .$$

4. NUMERICAL RESULT

In this section we shall present a numerical example :

$$-u'' = -e^u , \quad x \in [0,1], \quad u(0) = u(1) = 0.$$

This problem is considered in |3| and |6|, and its exact solution is

$$u(x) = -\ell n2 + 2 \,\ell n(c \sec \frac{c(x-0.5)}{2}) , \quad c = 1.3360557 \ldots .$$

For $x \in [0,1]$ is $-1 \leq u(x) \leq 0$.

To compute the approximation of $u(x)$ we define (see |2|)

$$f(x,v) = \begin{cases} -e^{-1} , & \text{for } v \leq -1, \\ -e^v , & \text{for } -1 \leq v \leq 0 , \\ -1 , & \text{for } v \geq 0, \end{cases}$$

satisfying (L) with $q = e^{-1} - 1$, $\mu = 0$. Using our first scheme, with $n = 20$, we iterate according to

$$y^o = [1,1,\ldots,1]^T \in \mathbb{R}^{42}, \quad A_h y^m = F_h y^{m-1}, \quad m \in \mathbb{N} .$$

From Theorem 7 (see |2|) it follows that

$$|y^m - y| \leq A_h^{-1} |A_h y^m - F_h y^m| , \quad m \in \mathbb{N} ,$$

where y is the solution of (DBVP1) for our example. Also it holds that

$$|y^m - y| \leq w^m , \quad m \in \mathbb{N} ,$$

where w^m is a solution of $A_h w^m \geq |A_h y^m - F_h y^m|$, $m \in \mathbb{N}$.

We have calculated in double precision arithmetic on PDP 11/340.

The numerical results are

$$h = 0.18740391 \cdot 10^{-1}, \ m = 16, \ \| y^m - y \|_\infty < 10^{-16},$$

$$\| y^m - u^h \|_\infty \leq 10^{-7} \ ,$$

where u^h denotes the restriction of u to the mesh I_h.

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REZIME

O NUMERIČKOM REŠAVANJU KONTURNOG PROBLEMA
POMOĆU OPTIMALNOG NUMERIČKOG DIFERENCIRANJA

U radu se posmatra numeričko rešavanje konturnih problema (BVP1) i (BVP2) pod pretpostavkama (L), (1), (2) osnosno (3). Za formiranje diskretnih analogona (DBVP1) i (DBVP2) koriste se optimalne četvorotačkaste formule reda 3 za numeričku aproksimaciju drugog izvoda iz |4| i |5|. Pri tom se koriste specijalne mreže diskretizacije I_h i J_k. Dokazano je da pod navedenim pretpostavkama opisani diferencni postupci imaju red konvergencije 3 i da postupak paralelne sečice za rešavanje (DBVP1) konvergira. Numerički je rešen primer iz |3| i |6|.