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SOME ALGEBRAIC PROPERTIES OF REGULAR MATROIDS

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ABSTRACT

The regular matroids mark an interesting half-way stage between the matroids corresponding to graphs on the one hand, and the binary matroids, that is matroids which are representable over GF(2), on the other. Perhaps the most famous result to date in all of matroid theory is Tutte's characterization of regular matroids by means of forbidden minors |2|. An interesting feature of regular matroids is their close relationship with an important class of matrices, the unimodular matrices |4| (note that the entries of a unimodular matrix are all 0 or ± 1).

Our aim in this paper is to give some algebraic properties of the standard representatives matrices of regular matroids.

INTRODUCTION

The matroid theoretic terminology and results used in this paper are according to standard literature (e.g., see |2, 4,5|). Let E be a finite set and r a function $r : 2^E \to \mathbb{N}$ (2^E is the power set of E and $\mathbb{N}$ the set of non-negative integers). Then the pair (E,r) is a <u>matroid</u> $M := M(E,r)$ on E, and $r(S)$ is the <u>rank</u> of $S \subseteq E$, if the following conditions hold:

(a) $r(S) \leq |S|$, for each $S \subseteq E$,

(b) if $S \subseteq S' \subseteq E$, then $r(S) \leq r(S')$,

(c) $r(S \cup S') + r(S \cap S') \leq r(S) + r(S')$, for each $S, S' \subseteq E$.

A subset $S \subseteq E$ is called <u>independent</u> if $r(S) = |S|$ where $|S|$ denotes the cardinality of S; a <u>basis</u> of M is a maximal independent subset of E.

If B is a basis of M, then $B^* = E - B$ is called a <u>cobasis</u> of M. The <u>dual</u> matroid of M is the matroid M^* on E whose bases are the cobases of M. If r^* is the rank function of M^*, then $r^*(S) = |S| - r(E) + r(E-S)$ for every $S \subseteq E$. Let $F(M)$ be the family of independent sets of M, and $\mathbb{F}$ a field. M is <u>representable</u> |1| over $\mathbb{F}$ if there exists a vector space V over $\mathbb{F}$ and an injection $\sigma : E \to V$ such that a subset S of E belongs to $F(M)$ if and only if the corresponding vectors of $\sigma(S)$ are linearly independent over $\mathbb{F}$. A matroid is <u>regular</u> |4| if it is representable over any field.

Throughout, we shall denote $r := r(E)$; it is well-known that for any basis B of M we have $r(B) = r(E)$. Let then B be a basis of M and $m = |E-B|$. If M is representable over a field $\mathbb{F}$ it will have a <u>standard matrix representation</u> |2,4| with respect to the basis B of the form $R(M,B) = [I_r | A]$ where I_r is the $r \times r$ identity matrix and A is an $r \times m$ matrix with the entries belonging to $\mathbb{F}$.

A well-known property of matroid representation (stated by Tutte in |2|) says that if M has a standard representation $R(M,B) = [I_r | A]$, then the dual M^* has a standard representation $R^*(M,B^*) = [-A^T | I_m]$ where A^T is the transposed of A and $B^* = E - B$. The following hold:

$$(1) \qquad [R(M,B)]\,[R^*(M,B^*)]^T = O_{rm},$$

$$(2) \qquad [R^*(M,B^*)]\,[R(M,B)]^T = O_{mr},$$

where O_{pq} denotes the null matrix with p rows and q columns.

<u>The main results</u>. In the sequel we shall consider M to be a regular matroid on the finite set $E = \{e_1, e_2, \dots, e_n\}$. If S is any subset of E and R a standard representation matrix of M we define $R(S)$ as the submatrix of R consisting of those columns that correspond to members of S.

THEOREM 1. *(W.T.Tutte, |3|). The matrices* R(M,B) *and* $R^*(M,B^*)$ *are unimodular.*

THEOREM 2. *(W.T.Tutte, |3|). Let* S *be a subset of* E. *The determinant of* R(S) *has one of the values* 1 *or* −1 *if* S *is a basis of* M *and* 0 *otherwise.*

THEOREM 3. *(W.T.Tutte, |3|). Let* S *be a subset of* E. *The determinant of* R*(S) *has one of the values* 1 *or* −1 *if* S *is a cobasis of* M *and* 0 *otherwise.*

COROLLARY *(W.T.Tutte, |3|). The following hold:*

(3) $$\det([R(M,B)][R(M,B)]^T) = b(M) ,$$

(4) $$\det([R^*(M,B^*)][R^*(M,B^*)]^T) = b(M) .$$

P r o o f. It follows from Theorem 1 and 2 by applying the Binet-Cauchy theorem:

$$\det([R(M,B)][R(M,B)]^T) = \sum_{i=1}^{b(M)} [\det R(B_i)]^2 = \sum_{i=1}^{b(M)} 1 = b(M),$$

where B_i, $i=1,2,\dots,b(M)$ are the bases of M. Similarly we can prove (4). (Q.E.D.)

LEMMA 1. *For any* r x m *matrix* A *the following hold:*

$$\det(I_r + AA^T) \geq 0.$$

P r o o f. Let X be an arbitrary r x 1 vector. Thus $X^T(I_r + AA^T)X = X^TX + X^TAA^TX = X^TX + (A^TX)^T(A^TX) = \|X\|^2 + \|A^TX\|^2 \geq 0$, i.e., $I_r + AA^T$ is a positive definite matrix ($\|X\|$ denotes the norm of X). Since $I_r + AA^T$ is symmetrical it follows by Sylvester's criterion (e.g., see |6|) that $\det(I_r + AA^T) \geq 0$. (Q.E.D.)

LEMMA 2. |6| *If* Y *is a square matrix of the form* $Y = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *where* A *and* D *are square matrices, then the following hold:*

(a) *if* $\det A \neq 0$, *then* $\det Y = \det A \det(D - CA^{-1}B)$,

(b) *if* $\det D \neq 0$, *then* $\det Y = \det D \det(A - BD^{-1}C)$.

LEMMA 3. *The following hold:*

(c) $$\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \geq 0 ,$$

(d) $$\det\left[\frac{R^*(M,B^*)}{R(M,B)}\right] = (-1)^{mr} \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] .$$

P r o o f. According to Lemma 2 we have

$$\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] = \det\left[\begin{array}{c|c} I_r & A \\ \hline -A^T & I_m \end{array}\right] = \det(I_m + A^TA) = \det(I_r + AA^T),$$

and (c) follows from Lemma 1. On the other hand, by Lemma 2, we have

$$\det\left[\frac{R^*(M,B^*)}{R(M,B)}\right] = \det\left[\begin{array}{c|c} -A^T & I_m \\ \hline I_r & A \end{array}\right] =$$

$$= (-1)^{mr}\det\left[\begin{array}{c|c} I_m & -A^T \\ \hline A & I_r \end{array}\right] = (-1)^{mr}\det (I_r + AA^T) =$$

$$= (-1)^{mr}\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] . \quad \text{(Q.E.D.)}$$

THEOREM 4. *The following hold:*

(5) $$\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] = b(M)$$

(6) $$\det\left[\frac{R^*(M,B^*)}{R(M,B)}\right] = \pm\, b(M) .$$

P r o o f. From (1) - (4) we obtain:

$$\left(\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right]\right)^2 = \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] =$$

$$= \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \det\left[[R(M,B)]^T \middle| [R^*(M,B^*)]^T \right] =$$

$$= \det\left(\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \left[[R(M,B)]^T \middle| [R^*(M,B^*)]^T \right] \right) =$$

$$= \det\left[\begin{array}{c|c} [R(M,B)][R(M,B)]^T & [R(M,B)][R^*(M,B^*)]^T \\ \hline [R^*(M,B^*)][R(M,B)]^T & [R^*(M,B^*)][R^*(M,B^*)]^T \end{array}\right] =$$

$$= \det\left[\begin{array}{c|c} [R(M,B)][R(M,B)]^T & O_{rm} \\ \hline O_{mr} & [R^*(M,B^*)][R^*(M,B^*)]^T \end{array}\right] =$$

$$= (-1)^{2\frac{r(r+1)}{2}} \det([R(M,B)][R(M,B)]^T)\det([R^*(M,B^*)][R^*(M,B^*)]^T) =$$

$$= [b(M)]^2 .$$

Thus, (5) follows according to (c) and (6) follows by (5) and (d). (Q.E.D.) Let B be a fixed basis of M, $B^* = E - B$ and B_i, $B_i^* = E - B_i$, $i=1,2,\dots,b(M)$. For every B_i we denote by $s(B_i)$ the sum of the columns´ indices of $R(M,B)$ that correspond to members of B_i. Similarly for $s(B_i^*)$. By expansion of $\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right]$ according to Laplace´s rule considering all major square submatrices of order r contained in the rows of $R(M,B)$, and using Theorem 2 and 3 we obtain from (5):

$$S = \sum_{i=1}^{b(M)} (-1)^{s(B_i)} \det R(B_i) \det R^*(B_i^*) = b(M) .$$

Similarly, using $R^*(M,B^*)$, by (6) we have:

$$S^* = \sum_{i=1}^{b(M)} (-1)^{s(B_i^*)} \det R(B_i) \det R^*(B_i^*) = \pm\, b(M) .$$

Obviously, $S = b(M)$ if and only if $(-1)^{s(B_i)} \det R(B_i) \det R^*(B_i^*) = 1$ for every $i=1,2,\dots,b(M)$, i.e. if and only if $\det R(B_i) = (-1)^{s(B_i)} \det R^*(B_i^*)$ for every $i=1,2,\dots,b(M)$. Similarly $S^*=b(M)$ if and only if $\det R(B_i) = (-1)^{s(B_i^*)} \det R^*(B_i^*)$ for every $i=1,2,\dots,b(M)$. On the other hand, $S^* = -b(M)$ if and only if $(-1)^{s(B_i^*)} \det R(B_i) \det R^*(B_i^*) = -1$ for every $i=1,2,\dots,b(M)$, i.e., if and only if $\det R(B_i) = (-1)^{s(B_i^*)+1} \det R^*(B_i^*)$ for every $i=1,2,\dots,b(M)$. Thus we have the following theorems:

THEOREM 5. *The following holds:*

$$\det R(B_i) = (-1)^{s(B_i)} \det R^*(B_i^*), \quad i=1,2,\ldots,b(M).$$

THEOREM 6. *Only one of the following holds:*

either $\det R(B_i) = (-1)^{s(B_i^*)} \det R^*(B_i^*), \; i=1,2,\ldots,b(M),$

or $\det R(B_i) = (-1)^{s(B_i^*)+1} \det R^*(B_i^*), \; i=1,2,\ldots,b(M).$

Let B' be a basis of M such that $B \neq B'$, $B - B' = \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\}$ and $B' - B = \{e_{j_1}, e_{j_2}, \ldots, e_{j_t}\}$ (obviously $|B-B'| = |B'-B|$). We shall denote by $R(B-B', B'-B)$ the square submatrix of R(M,B) consisting of elements in the crossing of the rows $i_1, i_2, \ldots, i_t$ with the columns $j_1, j_2, \ldots, j_t$.

THEOREM 7. *The following holds:*

$$\det R(B-B', B'-B) = \det R(B').$$

P r o o f. Obviously, the columns t+1, t+2,...,r of R(M,B) have only one non-null entry, i.e.,

$$(7) \quad (R(M,B))_{t+1,t+1} = (R(M,B))_{t+2,t+2} = \ldots = (R(M,B))_{r,r} = 1.$$

Let Δ_{t+1} denote the subdeterminant of $\det R(B')$ obtained by deleting the row t+1 and the column t+1: Δ_{t+2} the subdeterminant of Δ_{t+1} obtained by deleting the row t+2 and the column t+2 and so on up to Δ_r. From (7) and the definition of R(M,B) it follows that $\det R(B') = \Delta_{t+1} = \Delta_{t+2} = \ldots = \Delta_r = \det R(B-B', B'-B)$. (Q.E.D.).

Similarly, if $B^{*\prime}$ is a cobasis of M such that $B^{*\prime} \neq B^*$ we then have:

THEOREM 8. *The following holds:*

$$\det R^*(B^* - B^{*\prime}, B^{*\prime} - B^*) = \det R^*(B^{*\prime}).$$

Let B_a, B_b be two distinct bases of M, B_a^*, B_b^* their corresponding cobases and R_a, R_b, R_a^*, R_b^* the respectively standard representative matrices. Let $d(B_a,B_b) = |B_a-B_b| = |B_b-B_a|$. According to the form of R and R* and since $B_a-B_b = B_b^*-B_a^*$, $B_b-B_a = B_a^*-B_b^*$ we then have

$$\text{(8)}\quad \det R_a(B_a-B_b,\ B_b-B_a) = (-1)^{d(B_a,B_b)}\det R_a^*(B_a^*-B_b^*,B_b^*-B_a^*)\ ,$$

$$\text{(9)}\quad \det R_b(B_b-B_a,B_a-B_b) = (-1)^{d(B_a,B_b)}\det R_b^*(B_b^*-B_a^*,B_a^*-B_b^*)\ .$$

From (8) and (9), using Theorems 7 and 8 we obtain:

$$\text{(10)}\quad \det R_a(B_b) = (-1)^{d(B_a,B_b)}\det R_a^*(B_b^*)\ ,$$

$$\text{(11)}\quad \det R_b(B_a) = (-1)^{d(B_a,B_b)}\det R_b^*(B_a^*)\ .$$

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REZIME

NEKE ALGEBARSKE OSOBINE REGULARNIH MATROIDA

U ovom radu dokazane su neke algebarske osobine standardnih reprezentativnih matrica regularnih matroida.