Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 12 (1982)
Review of Research Faculty of Science-University of Novi Sad, Volume 12(1982)

COUNTING BINARY GRIDS

Ratko Tošić and Vojislav Petrović
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

ABSTRACT

A binary (m,n)-grid is defined as an array of m rows and n columns formed from mn square cells each of which is crossed by a diagonal. Two (m,n)-grids are said to be equivalent iff they can be transformed one into the other by rigid motion in the space. In this paper the number N(m,n) of non-equivalent (m,n)-grids are determined for arbitrary natural numbers m and n. The formula of N.Hoffman for N(1,n) appears as a special case of our result.

1. INTRODUCTION

A binary (m,n)-grid is defined as an array of m rows and n columns formed from mn square cells each of which is crossed by a diagonal. Thus, Fig. 1 is a (3,4)-grid.

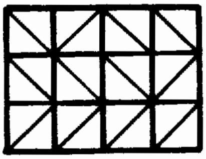


Fig. 1.

Two (m,n)-grids are said to be equivalent iff they can be transformed one into the other by a rigid motion in space.

J.A. Dunn posed the problem of determining the number N(m,n) of non-equivalent (m,n)-grids. Clearly, N(m,n) = N(n,m). N.Hoffman in |1| proved that

$$N(1,n) = 2^{n-2} + 2^{\left\lfloor\frac{n}{2}\right\rfloor - 1} . \quad (1)$$

In this paper we shall determine the number $N(m,n)$, for arbitrary natural numbers m and n; the Hoffman's formula appears as a special case of our result.

Let us introduce some notations. By $Y_{(m,n)}$ we shall denote the set of all (m,n)-grids. Once m and n are specified, we shall write simply Y instead of $Y_{(m,n)}$. If A is a set, then the cardinality of A is $|A|$. By $\lfloor x \rfloor$ and $\lceil x \rceil$ we shall denote the smallest integer $\geq x$ and the greatest integer $\leq x$, respectively. It is clear that

$$|Y_{(m,n)}| = 2^{mn} \quad . \tag{2}$$

2. RECTANGULAR (m,n)-GRIDS ($m \neq n$)

Any rigid motion in space by which an (m,n)-grid can be transformed into some other (may be the same) (m,n)-grid - reduces to one of the transformations from the group of symmetries of the rectangle (Fig. 2):

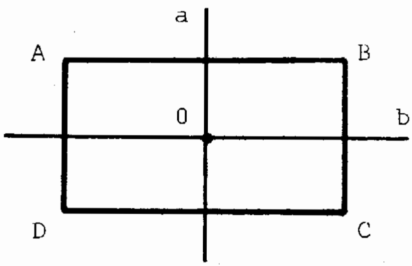


Fig. 2.

i - identical transformation,

a - symmetry with respect to the vertical axis,

b - symmetry with respect to the horizontal axis,

c - symmetry with respect to the center 0 of the rectangle (central symmetry).

Let $t(X)$ denote the grid into which the grid X is transformed by applying the transformation t. We shall consider the following subset of X.

$$A = \{X / a(X) = X\},$$
$$B = \{X / b(X) = X\},$$
$$C = \{X / c(X) = X\} .$$

Using Burnside's lemma, we obtain:

LEMMA 2.1. *If* $m \neq n$, *then*

(3) $$N(m,n) = \frac{1}{4}(|Y| + |A| + |B| + |C|).$$

3. SQUARE GRIDS ((n,n)-GRIDS)

In addition to i, a, b and c, the group of rigid motions which transforms a square into itself, contains four additional transformations (Fig. 3):

d - rotation about the center 0 of square through angle $\phi = 90^{\circ}$,

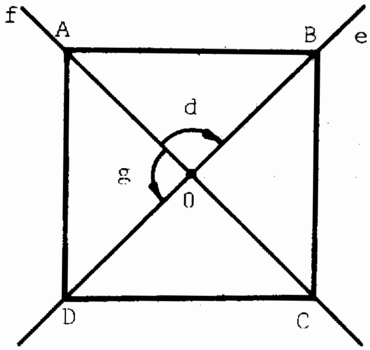


Fig. 3.

e - symmetry with the respect to the diagonal BD,

f - symmetry with the respect to the diagonal AC,

g - rotation about the center 0 through angle $\phi = -90^{\circ}$.

Let A,B,C be subsets of $Y = Y_{(n,n)}$, defined as in Section 2. We shall consider also the following subsets of X:

$$D = \{X/d(X) = X\};$$
$$F = \{X/f(X) = X\};$$
$$G = \{X/g(X) = X\}.$$

Using Burnside's lemma, we obtain:

LEMMA 3.1.

(4) $$N(n,n) = \frac{1}{4}|D| + \frac{1}{8}(|Y|+|A|+|B|+|C|+|E|+|F|).$$

4. THE THEOREM

Before stating the theorem, we shall prove some lemmas.

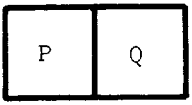
LEMMA 4.1. *For arbitrary natural* m,n, *and for the corresponding subsets* A,B,C *of* $Y_{(m,n)} = Y$

$$(5) \qquad (i) \qquad |A| = (1+(-1)^n)\,2^{\frac{mn}{2}-1},$$

$$(6) \qquad (ii) \qquad |B| = (1+(-1)^m)\,2^{\frac{mn}{2}-1},$$

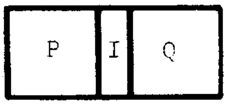
$$(7) \qquad (iii) \qquad |C| = 2^{\lceil \frac{mn}{2} \rceil}.$$

P r o o f. (i) For $n = 2k$ $(k \in N)$, an arbitrary grid X from $Y_{(m,n)}$ can be represented in the form



where P and Q are (m,k)-grids. Now, $X \in A$ iff $Q = a(P)$. It means that $X \in A$ is determined by its (m,k)-subgrid P (containing $mk = \frac{mn}{2}$ cells), hence follows the statement.

For $n = 2k+1$ $(k \geq 0)$, an arbitrary grid X from $Y_{(m,n)}$ can be represented in the form



where P and Q are (m,k)-grids and I is a (m,1)-grid. Now, $X \in A$ iff $Q = a(P)$ and $I = a(I)$. However, $a(I) \neq I$, hence $A = \emptyset$, i.e. $|A| = 0$, and (5) is proved.

(ii) (6) can be proved in the same way.

(iii) $X \in C$ iff any two cells situated symmetrically with respect to the center 0 of X are of the same sort (either ⧄ or ⧅). If mn is even, $X \in C$ is determined by $\frac{mn}{2} = \lceil \frac{mn}{2} \rceil$ pairs of cells. If both m and n are odd, the central cell is symmetrical to itself and it can be of an arbitrary sort. In that case, $X \in C$ is determined by $\frac{mn-1}{2} + 1 = \lceil \frac{mn}{2} \rceil$ pairs of cells (central cell included). Hence, follows the statement.

REMARK.

If $m = n$, (5), (6) and (7) become:

(8) $$|A| = |B| = (1 + (-1)^n)\, 2^{\frac{n^2}{2} - 1}$$

(9) $$|C| = 2^{\frac{n^2}{2}}$$

LEMMA 4.2. *For arbitrary* $n \in N$, *and for corresponding subsets* D, E, F of $Y_{(m,n)} = Y$:

(10) (i) $$|E| = |F| = 2^{\frac{n^2+n}{2}} ,$$

(11) (ii) $$|D| = (1 + (-1)^n)\, 2^{\frac{n^2}{4} - 1} .$$

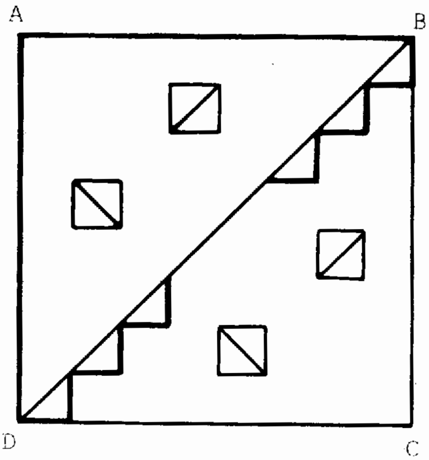
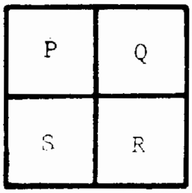


Fig. 4.

P r o o f. (i) $X \in E$ iff any two cells, situated symmetrycally with respect to the diagonal BD (Fig. 4), are of the same sort (either ⧄ or ⧅). Each cell situated on the diagonal BD may be of an arbitrary sort (either ⧄ or ⧅). This means that $X \in E$ is determined by its cells situated on the diagonal BD and above it. The number of such cells is 1+2+... $\dots + n = \frac{n^2+n}{2}$. Hence it follows that $|E| = 2^{(n^2+n)/2}$.

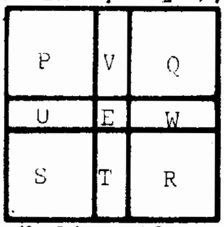
In the same way we can prove that $|F| = 2^{(n^2+n)/2}$.

(ii) If n is even ($n = 2k$, $k \geq 1$), then $X \in Y_{(n,n)}$ can be represented in the form



where P, Q, R and S are (k,k)-grids. Now, $X \in D$ iff $Q = d(P)$, $R = d(Q) = d^2(P)$ and $S = d(R) = d^3(P)$. This means that $X \in D$ is determined by its subgrids P containing $n^2/4$ cells, hence follows the statement.

If n is odd ($n = 2k+1$, $k \geq 0$), then $X \in Y_{(n,n)}$ can be represented in the form



where P, Q, R and S are (k,k)-grids, U and W - (1,k) - grids, V and T - (k,1)-grids and E - (1,1) -grid. Now $X \in D$ implies $d(E) = E$, which is a contradiction, because d(⧄) = ⧅ and d(⧅) = ⧄ . This means that if n is odd, then $D = \emptyset$, i.e. $|D| = 0$, and (11) is proved.

Now, we can prove the theorem.

THEOREM. *(i) If* $m \neq n$ *, then*

(12) $$N(m,n) = 2^{mn+2} + 2^{\frac{mn-4}{2}} + 2^{\left\lceil\frac{mn-4}{2}\right\rceil} + ((-1)^m + (-1)^n)2^{\frac{mn-6}{2}} ;$$

(ii) $$N(n,n) = 2^{n^2-3} + 2^{\left\lceil\frac{n^2-6}{2}\right\rceil} + 2^{\frac{n^2+n-4}{2}} +$$

(13) $$+ (1 + (-1)^n)\left(2^{\frac{n^2-12}{4}} + 2^{\frac{n^2-6}{2}}\right) .$$

P r o o f. (i) Follows from (3), (2), (5), (6) and (7).

(ii) Follows from (4),(2),(8),(9),(10) and (11).

REMARK

For $m=1$, (12) reduces to Hoffman´s formula (1).

REFERENCES

|1| N.Hoffman, *Binary grids and a related counting problem*, *Two-Year Coll. Math. J.* 9(1978) 267-272.

Received by the editors February 10,1983.

REZIME

PREBROJAVANJE BINARNIH REŠETKI

Binarna (m,n)-rešetka definiše se kao pravougaonik podeljen na mn podudarnih kvadrata, koji su rasporedjeni u m vrsta i n kolona, pri čemu je svaki od njih presečen jednom dijagonalom. Dve binarne rešetke su ekvivalentne ako se kretanjem u prostoru mogu dovesti do poklapanja.

U radu je odredjen broj N(m,n) neekvivalentnih (m,n)-rešetki, za proizvoljne prirodne brojeve m i n, čime je uopštena formula N. Hoffmana za N(1,n).