Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 12 (1982)
Review of Research Faculty of Science-University of Novi Sad, Volume 12(1982)

A NOTE ON FORCING AND WEAK INTERPOLATION THEOREM FOR INFINITARY LOGICS

Milan Grulović
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića br.4, Jugoslavija

ABSTRACT

Our considerations are connected with the results of the first three chapters of |1|. The aim of this paper is to contrubute in some way to a better understanding of the purpose of introducing weak formulas while dealing with the forcing relation for infinitary logics (Theorem 1.23) as well as to correct, in our opinion, the proof of the weak form of the Interpolation theorem for infinitary logics.

INTRODUCTION

It is already announced in |1| that the proofs of: for each p $p \| \!\vdash \sim\sim PC_\infty 1$ or of preserving E_∞ "seems to involve some kind of saturation property for C" i.e. (*) for each p from $p \| \!\vdash \bigwedge \sim\sim \Phi$ follows $p \| \!\vdash \sim\sim \bigwedge \Phi$ (we are always given the other implication). We have shown that all these statements are in fact equivalent (thus mutually equivalent) to: for each p (and ϕ) $p \| \!\vdash \sim\sim \phi$ iff $p \| \!\vdash \phi^{wk}$ (Theorem 1.23). A sufficient condition, merely conjectured in |1|, that these statements hold, is that any nondecreasing sequence of length $\alpha < k$, when a fragment of some logic $L_{k\mu}$ is considered, has an upper bound (Lemma 1.11). But this is not a necessary condition too (example 1.13). From (*) follows also: for each p $p \| \!\vdash \sim\sim PC11$ which otherwise when the given logic is infinitary does not have to be fulfilled (Example 1.15).

As for the proof of the Weak Interpolation Theorem for infinitary logics (semantical $\models$ is replaced by syntactical $\vdash$) our main objection is that relation ($\Vdash$) applied in it does not have to be (and in cases of real interest, is not) a forcing relation, while on the other side the properties of forcing relations, in particular 1.22, are used. All the troubles are overcome successfully by the construction of a forcing relation which has a "nice" intersection with the given one (Lemmas 2.12 - 2.23, Theorem 2.24).

Some other corrections and a few, we hope, useful remarks are made.

§ 0. We shall assume a knowledge of the basic properties of a forcing relation and in particular a familiarity with |1|. However for the reader's convenience we shall cite some of the most relevant definitions and results, mostly from |1|, maybe with some slight, unessential reformulations but using the same terminology and notation.

Through the whole article the language L in question, in all general discussions, will be a first-order language (finitary or not) containing at least one constant symbol. The basic logic symbols will be $\sim$ (negation), & (conjunction), $\exists$ (existential quantifier) and (in the case of infinitary logics) Λ (infinite conjunction). The other like $\vee$ (disjunction), $\rightarrow$ (implication), $\forall$ (universal quantifier) and V (infinite disjunction) are defined by the basic ones in the standard way. $\Lambda\Phi$ will replace $\bigwedge_{\phi\in\Phi}\phi$.

We shall just recall that the system of axioms for finitary logic used in |1|, is divided into the groups: (A) substitution instances of propositional tautologies, (B) basic quantificational axioms, (D) generalized quantificational axioms and when the logic is with equality (C) identity axioms. The only rule of inference is modus ponens. For the necessity of infinitary logics we shall redefine in the natural way the axioms of group (D) (and get D_∞), add a new set of axioms (A_∞) $PC_\infty 1 : \Lambda\sim\sim\Phi\rightarrow\Lambda\phi$, $PC_\infty 2 : \Lambda\Phi\rightarrow\phi$ for all $\phi\in\Phi$ and one more rule of inference E_∞ : if $\psi\rightarrow\phi$ for all $\phi\in\Phi$ then $\psi\rightarrow\Lambda\Phi$. Of course, in the formulas of other axiom schemes, formulas of infinite length can occur. One can easily see that there a redundancy in the offered system of axioms. So, for instance part (A) of it can be rather restricted and it is obvious that $PC_\infty 2$ and E_∞ with the help of the axioms of part (A) give $PC_\infty 1$. The set of all these axioms and rules of inference will be denoted by Λ_o.

§ 1. Let $\langle C, \leq, 0\rangle$ be a partially ordered set with the least element 0, AT(L) the set of atomic and SENT(L) the set of all sentences of a language L (we will often write only AT and SENT rather than AT(L) and SENT(L) on the condition that it is clear from the context what is meant by it

DEFINITION 1.1. *A unary relation* $\Vdash$ *on* C x SENT(L) *is a forcing relation if it satisfies the following conditions (instead of* $(p,\phi) \in \Vdash$ *we shall use the more common* $p \Vdash \phi$ *; of course* $p \nVdash \phi$ *will stand for* $(p,\phi) \notin \Vdash$ *):*

(1) The compability condition (s): *:*
For each $p, q \in C$ *, for any* $\phi \in AT$
$p \leq q$ *and* $p \Vdash \phi$ *imply* $q \Vdash \phi$ *;*

If L *is the language with equality we demand also*

(1) (i) For each $p \in C$ *and each closed term* t *there exists* $q \in C$, $q \geq p$ *and* $q \Vdash t = t$.

(ii) For all closed terms t_1, t_2 *, for any atomic formula* $\phi(v)$ *with at most one variable free and for each* $p \in C$ *there exists* $q \in C$ *such that* $q \geq p$ *and either* $p \nVdash t_1 = t_2$ *or* $p \nVdash \phi(t_1)$ *or* $q \Vdash \phi(t_2)$*;*

(2) $p \Vdash \phi_1 \& \phi_2$ *if and only if* $p \Vdash \phi_1$ *and* $p \Vdash \phi_2$ *;*

If L *is an infinitary language we introduce too*

(2) (i) $p \Vdash \Lambda\Phi$ *if and only if* $p \Vdash \phi$ *for each* $\phi \in \Phi$ *;*

(3) $p \Vdash \sim\phi$ *if and only if for each* $q \geq p$ $q \nVdash \phi$

(4) $p \Vdash \exists v\phi(v)$ *if and only if there exists a closed term* t *such that* $p \Vdash \phi(t)$.

The elements of C will be, as usual, called conditions. We read $p \Vdash \phi$ as p forces ϕ . When $p \Vdash \sim\sim\phi$ we say that (a condition) p weakly forces ϕ.

In defining some forcing relation we shall give only its intersection with $C \times AT$ which is obviously sufficient.

DEFINITION 1.2 *A forcing system is a triple* $\langle C, \Vdash, L\rangle$ *where* C *is a partially ordered set with the least element*, L *a given language and* $\Vdash$ *a forcing relation on* $C \times SENT(L)$.

DEFINITION 1.3. *Let* $\langle C, \Vdash, L\rangle$ *be a forcing system where* L *is a finitary logic. For* $p \in C$

$$T^C|p| = \{\phi \in SENT | p \Vdash \sim\sim \phi\}$$

Instead of $T^C|0|$, *where* 0 *is the least element of* C *we write just* T^C. T^C *is called the (forcing) companion.*

In any of the propositions that follow, if it is not already written it goes without saying that some forcing system $\langle C, \Vdash, L\rangle$, fixed but without any special characteristics, is given.

The following assertions are direct consequences of definition 1.1.

THEOREM 1.4. *(a) For any conditions* p,q *and for any sentence* ϕ *if* $p \Vdash \phi$ *and* $q \geq p$ *then* $q \Vdash \phi$;

(b) For each $p \in C$ *and for each sentence* ϕ *either* $p \not\Vdash \phi$ *or* $p \not\Vdash \sim\phi$;

(c) For each $p \in C$ *and for each sentence* ϕ *there exists a condition* $q \geq p$ *such that* $q \Vdash \phi$ *or* $q \Vdash \sim \phi$.

LEMMA 1.5. *(a) If* $p \Vdash \phi$ *then* $p \Vdash \sim\sim \phi$

(b) $p \Vdash \sim \phi$ *if and only if* $p \Vdash \sim\sim\sim \phi$

(c) $p \Vdash \sim\sim \& \Phi$ *if and only if* $p \Vdash \& \sim\sim \Phi$

(d) $p \Vdash \sim \exists v \sim \phi$ *if and only if for all closed terms* t $p \Vdash \sim\sim \phi(t)$.

In |1| is given a complete and very exhaustive syntactic proof of

THEOREM 1.6. *Let* $\langle C, \Vdash, L\rangle$ *be a forcing system where* L *is a finitary language (of an arbitrary cardinality). Then for each* $p \in C$

(1) $T^C|p|$ *is a consistent, deductively closed set* ($T^C|p| \vdash \phi$ *implies* $\phi \in T^C|p|$);

(2) *If* $\phi(v_1,\ldots,v_n)$ *is a logically valid formula (i.e.* $\vdash_L \phi(v_1,\ldots,v_n)$) *then for any closed terms* $t_1,\ldots,t_n$ $\phi(t_1,\ldots,t_n) \in T^C|p|$.

In this place we would like to mention two things. Firstly, (any) condition p really forces, not merely weakly forces, each of the axioms (for a finitary logic). This follows from Lemmas 1.5 and

LEMMA 1.7. $p \Vdash \sim(\sim\sim\phi \,\&\, \psi)$ *if and only if* $p \Vdash \sim(\phi \,\&\, \psi)$ *which give.*

LEMMA 1.8. $p \Vdash \sim\sim(\phi \to \psi)$ *if and only* $p \Vdash \phi \to \psi$.

That is not a property of for instance the forcing relation defined in |4| where both & and V are taken for basic logic symbols and a part of the definition of a forcing relation is

$p \Vdash \phi \vee \psi$ if and only if either $p \Vdash \phi$ or $p \Vdash \psi$

while we have

LEMMA 1.9. *There exists* $q \geq p$, $q \Vdash \phi \vee \psi$ *if and only if there exists* $q \geq p$, $q \Vdash \phi$ *or* $q \Vdash \psi$.

Secondly, the result of Theorem 1.6. cannot be generalized i.e. an analogous assertion for infinitary logics with equality does not hold even if we kept "the syntactic apparatus" of the finitary logic possible enriched by $PC_\infty 2$. Namely it is easy to prove

LEMMA 1.10. *For all* $p \in C$, $p \Vdash \sim\sim PC_\infty 2$ *(that is* p *weakly forces any sentence belonging to the axiom scheme* $PC_\infty 2$ *)*

but let us try to check $p \Vdash \sim\sim\forall v \forall u(v = u \to (\phi \to \phi'))$ wher: ϕ is a formula in which u is free for v, ϕ' is the result o:

substituting some (not necessarily all) free occurences of v by u and ϕ is of the form $\wedge\Psi$ (infinite conjuction). In case of a finitary logic the proof, based on Definition 1.1, is given by induction on the complexity of ϕ.

First of all let us note that (for any ϕ)

$p\|\!\vdash \sim\sim\forall v \forall u (v = u \to (\phi \to \phi'))$ if and only if for all closed terms t_1, t_2 $p\|\!\vdash \sim(t_1 = t_2 \,\&\, \phi(t_1) \,\&\, \sim\phi'(t_1, t_2))$.

Let us suppose there exists a condition $q \geq p$ such that $q\|\!\vdash t_1 = t_2 \,\&\, \wedge\Psi \,\&\, \sim\wedge\Psi'$. By inductive hypothesis, from $q\|\!\vdash t_1 = t_2$ and $q\|\!\vdash \psi$, $\psi \in \Psi$ follows $q\|\!\not\vdash \sim\psi'$. Hence $q \|\!\not\vdash \sim\wedge\sim\sim\Psi'$ for in the opposite case for any $r \geq q$ there would exist a formula $\psi' \in \Psi'$ such that $q \|\!\not\vdash \sim\sim\psi'$. Let us fix such a pair r, ψ'. Then for some $s \geq r$ $s\|\!\vdash \sim\psi'$ while also $s\|\!\vdash t_1 = t_2$ and $s\|\!\vdash \psi$, contrary to the inductive assumption. Now the question is, whether $q \|\!\not\vdash \sim\wedge\sim\sim\Psi'$ is in contradiction with $q\|\!\vdash \sim\wedge\Psi'$. And the answer is that there is no a general answer, that is that the answer does not follow from the very definition of a forcing relation, as we shall soon see.

In case we wish to get an analogy of Theorem 1.6. for any infinitary logic, two of the possible ways to accomplish this are either to add some new assumptions to the set of conditions or to redefine the set $T^C|p|$ in the suitable way.

So if we have the $L_{k\omega}$ logic $(k > \omega)$ it holds (we assume the axiom of choice)

LEMMA 1.11. *If a partially ordered set of conditions* $\langle C, \leq \rangle$ *has the property that for each* $\lambda < k$ *any nondecreasing sequence* $p_0 \leq p_1 \leq \ldots \leq p_\alpha \leq \ldots, \alpha < \lambda$ *in* C *has an upper bound then:*

for each condition p *(and any set of sentences* Φ, $|\Phi| < k$*)*

(*) $p\|\!\vdash \wedge\sim\sim\Phi$ *if and only if* $p\|\!\vdash \sim\sim\wedge\Phi$

P r o o f. We shall consider only the less trivial implication. Let $\Phi = \{\phi_\gamma \mid \gamma < \lambda (< k)\}$, $p, q \in C$ and $p\|\!\vdash \wedge\sim\sim\Phi$ and let us suppose that for each $\alpha < \lambda$ we have already constructed a sequence $q \leq p_0 \leq \ldots \leq p_\beta \leq \ldots, \beta < \alpha$ so that $p_\beta\|\!\vdash \phi_\gamma$ for each $\gamma \leq \beta$.

If α is a limit ordinal just the assumption on partially ordered set enables us to extend the sequence with a new condition p_α ($p_\alpha \geq p_\beta$ for $\beta < \alpha$) which forces all formulas ϕ_γ, $\gamma \leq \alpha$ (in other words we can simply "bridge the gap " between successor and limit ordinals). The case that α is a successor ordinal is clear (compare with 1.5 (c)).

In the end, let us note only that the other (more trivial) implication always holds.

From (*) follows directly $p \Vdash \sim\sim \forall v \forall u (v = u \rightarrow (\phi \rightarrow \phi'))$ (thus, in general, in the notation from |1| $p \Vdash \sim\sim$ PC 11) also for infinitary logics because of

LEMMA 1.12. *The conditions "(*)" and "for each* p *$p \Vdash \sim \bigwedge \Phi$ if and only if $p \Vdash \sim \bigwedge \sim\sim \Phi$" are equivalent.*

P r o o f. An immediate consequence of Definition 1.1

However the condition on a partially ordered set from the previous lemma is not necessary in order that (*) holds. The following example shows this.

EXAMPLE 1.13. *Let* **M** *be an infinitely countable model of a countable language and let Δ be its diagram (the set of all atomic and negations of atomic sentences of the language* L(M) *which hold in the model M_M). Let us enumerate the sentences of $\Delta = \Phi_1 \cup \Phi_2$ where Φ_1 and Φ_2 are, respectively, the set of atomic, that is negations of atomic sentences from Δ in such a way that $\Phi_1 = \{\phi_n | n \in \omega\}$ and $\Phi_2 = \{\phi_{\omega+k} | k \in \omega\}$ Further let* $C = \{p_\alpha | \alpha < \omega + \omega\}$ *where $p_\alpha = \{\phi_\beta | \beta \leq \alpha\}$, be partially ordered by the inclusion relation and let us determine a forcing relation on* $C \times SENT(L(M)_{\omega_1\omega})$ *by:*

for atomic ϕ $p_\alpha \Vdash \phi$ if and inly if $\phi \in p_\alpha$

For the forcing system $\langle C, \Vdash, L(M)_{\omega_1\omega} \rangle$ (one can easily check that the given triple is really a forcing system) (*) holds i.e. for any condition $p = p_\alpha$ and any set Φ of sentences, $|\Phi| \leq \omega$

$$p \Vdash \sim\sim \bigwedge \Phi \quad \text{if and only if} \quad p \Vdash \bigwedge \sim\sim \Phi$$

Let us prove this. Since $\langle C, \subseteq \rangle$ is linearly ordered (with the least element $p_o = \{\phi_o\}$) for any β, γ ($<\omega+\omega$) and each sentence ϕ of the language $L(M)_{\omega_1\omega}$

$$p_\beta \Vdash \sim\sim \phi \quad \text{if and only if} \quad p_\gamma \Vdash \sim\sim \phi$$

Hence it is enough to check that

$$p_\omega \Vdash \sim\sim \wedge \Phi \quad \text{if and only if} \quad p_\omega \Vdash \wedge \sim\sim \Phi$$

Namely, on condition the assertion $p_\alpha \Vdash \sim\sim \wedge \Phi$ iff $p_\alpha \Vdash \wedge \sim\sim \Phi$ is true for at least one condition p_α (*) follows easily. For let $\beta \neq \alpha$ and $p_\beta \Vdash \wedge \sim\sim \Phi$. Then by the above $p_\alpha \Vdash \wedge \sim\sim \Phi$ whence $p_\alpha \Vdash \sim\sim \wedge \Phi$ and therefore $p_\beta \Vdash \sim\sim \wedge \Phi$

But for any sentence ϕ

$p_\omega \Vdash \phi$ if and only if $p_\omega \Vdash \sim\sim \phi$ (then clearly $p_\omega \Vdash \wedge \sim\sim \Phi$ implies that $p_\omega \Vdash \wedge \Phi$ and consequently $p_\omega \Vdash \sim\sim \wedge \Phi$).

The proof is by induction on the complexity of ϕ .

If ϕ is atomic and $p_\omega \Vdash \sim\sim \phi$ then $\phi \in \Phi_1$ and so $p_\omega \Vdash \phi$ since $\Phi_1 \subseteq p_\omega$.

If ϕ is $\exists v \psi(v)$ and $p_\omega \Vdash \sim\sim \exists v \psi(v)$ then for some $n \in \omega$ and some closed term t $p_{\omega+n} \Vdash \psi(t)$. Thus $p_\omega \Vdash \sim\sim \psi(t)$ and by the inductive hypothesis $p_\omega \Vdash \psi(t)$ i.e. $p_\omega \Vdash \exists v \psi(v)$.

Other cases are even more trivial.

EXAMPLE 1.14. *Let all suppositions, except the enumeration of* Δ *, be as in the previous case. New we shall suppose that*

$$\Delta = \{\phi_n \mid n \in \omega\} \quad \textit{(thus} \quad C = \{p_n \mid n \in \omega\}, \quad p_n = \{\phi_o, \dots, \phi_n\}).$$

It is easy to verify that for all $j \in \omega$

$$p_j \Vdash \wedge \sim\sim \Delta \quad \textit{and} \quad p_j \Vdash \sim \wedge \Delta$$

Thus in particular $p_j \not\Vdash \sim\sim \wedge \Delta$ *and we see that in this example of the forcing system (*) does not hold.*

We have only proved that (*) implies $T^C|p| \vdash$ PC 11 (i.e. for any ϕ which belongs to the axiom scheme PC 11 $T^C|p| \vdash \phi$ that is $\phi \in T^C|p|$). Thus in case (*) does not hold we have to check separately whether $T^C|p| \vdash$ PC 11 holds or not.

The next example, however, confirms our second remark given after 1.9.

EXAMPLE 1.15. *Let* **L** *be a language containing just equality relation = and let* **M** *be an infinitely countable model for* L *in which = is interpreted as an equivalence relation such that at least one equivalence class contains infinitely many elements. Let* **A** *be such a class and let* **a**, **b** *be two elements of* **A**. *Let* $\Delta = \Delta_1 \cup \Delta_2$ *be the positive diagram of* **M**, *where* Δ_1 *is the set of atomic sentences in which constant* c_a *appears, corresponding to the element a* ($\Delta_2 = \Delta \setminus \Delta_1$) *and let us enumerate the sentences of* Δ *so that* $\Delta_1 = \{\phi_n | n \in \omega\}$ *and* $\Delta_2 = \{\phi_{\omega+k} | k \in \omega\}$. *Again we put* $p_\alpha = \{\phi_\beta | \beta \leqslant \alpha\}$, $C = \{p_\alpha | \alpha < \omega+\omega\}$ *and define for the language* $L(M)_{\omega_1\omega}$ *a forcing relation as before.*

Now there is no one condition which would force weakly $c_a = c_b \rightarrow (\Lambda\, \Delta_1 \rightarrow \Lambda\, \Delta_1')$ *where* Δ_1' *is a result of the substitution of constant* c_a *by* c_b *in the sentences of* Δ_1, *for*

$$p_\omega \|\!\!- c_a = c_b \,\&\, \Lambda\, \Delta_1 \,\&\, \sim \Lambda\, \Delta_1'$$

In particular no condition forces weakly $\forall v \forall u (v = u \rightarrow (\Lambda \Delta_1(v) \rightarrow \Lambda\, \Delta_1'(u))$ *where* $\Delta_1'(v)$ *and* $\Delta_1'(u)$ *are obtained from* Δ_1 *that is* Δ_1' *by substituting constants* c_a, c_b *by, the variables* v *and* u *respectively.*

In |1| in order to obtain an analogous result to 1.6 for infinitary logics the notion of $T^C|p|$ is redefined. For that purpose firstly, the concept of "weak" formulas is introduced.

DEFINITION 1.16. *For a formula* ϕ *of (infinitary) logic* L *we define a "weak" formula* ϕ^{wk} *as follows:*

(i) *if* ϕ *is atomic* ϕ^{wk} *is* $\sim\sim\phi$;

(ii) *if* ϕ *is* $\Lambda\Psi$ ϕ^{wk} *is* $\Lambda\Psi^{wk}$ *(this case includes a finite conjuction);*

(iii) *if* ϕ *is* $\exists v \psi(v)$ ϕ^{wk} *is* $\sim\sim \exists v \psi^{wk}(v)$

and (iv) *if* ϕ *is* $\sim\psi$ ϕ^{wk} *is* $\sim\psi^{wk}$.

From the aspect of a forcing relation, as long as finitary logics are considered, nothing new is obtained by "weak" formulas because of

LEMMA 1.17. *If* $<C, ||\!\!-, L>$ *is a forcing system where* L *is a finitary language then for any condition* p *and any sentence* ϕ *of* L

$$p||\!\!-\sim\sim\phi \quad \text{and only if} \quad p||\!\!-\phi^{wk}$$

But independent of whether a logic L is finite or not it always holds:

LEMMA 1.18. $p||\!\!-\sim\sim\phi^{wk}$ *if and only if* $p||\!\!-\phi^{wk}$,

and hence

$$p||\!\!-\Lambda\sim\sim\Phi^{wk} \;\; iff \;\; p||\!\!-\Lambda\Phi^{wk} \;\; iff \;\; p||\!\!-\sim\sim\Lambda\Phi^{wk}.$$

So as we see "weak" formulas enable us to "draw out" the double negation in front of the infinite conjuction, more-over, to eliminate it. Now it follows directly from the conside-ration made after 1.10 and Lemma 1.12

LEMMA 1.19. $p||\!\!-(PC11)^{wk}$.

This lemma also follows from parts (a) and (b) of

LEMMA 1.20. *Let* $<C, ||\!\!-, L_{\lambda\mu}>$ *be a forcing system. Then (for any* $p \in C$ *):*

(a) *if* $p||\!\!-(\phi\to\psi)^{wk}$ *and* $p||\!\!-\phi^{wk}$ *then* $p||\!\!-\psi^{wk}$;

(b) $p||\!\!-(\Lambda\sim\sim\Phi\to\Lambda\Phi)^{wk}$ *(i.e.* $p||\!\!-(PC_\infty 1)^{wk}$) ;

(c) $p||\!\!-(\bigwedge_{\phi\in\Phi}(\psi\to\phi)\to(\psi\to\Lambda\Phi))^{wk}$ *whence:* $p||\!\!-(\psi\to\phi)^{wk}$ *for each* $\phi\in\Phi$ *implies* $p||\!\!-(\psi\to\Lambda\Phi)^{wk}$;

(d) *if* $\phi(v_1,\dots,v_\mu)$, $\mu<\lambda$ *is a quantificational formula for any closed terms* $t_1,\dots,t_\mu$ $p||\!\!-\phi^{wk}(t_1,\dots,t_\mu)$.

Thus for the generalized notion of $T^C|p|$ (see Lemma 1.17) given by

DEFINITION 1.21. *Let* $<C, ||\!\!-, L>$ *be a forcing system. For* $p \in C$

$$T^C|p| = \{\phi \in SENT \mid p||\!\!-\phi^{wk}\}$$

(again for $T^C|0|$ *we use only* T^C *)*

one obtains

THEOREM 1.22. |1| *Let* $\langle C, \|\!\!-, L\rangle$ *be a forcing system where* L *is an infinitary logic with the set of axioms and rules of inference* Λ_o. *Then for any* $p \in C$

(1) $T^C|p|$ *is a consistent deductively closed set ;*

(2) *if* $\phi(v_1,\ldots,v_\mu)$ *is formula of the language* L *and* $\vdash_L \phi(v_1,\ldots,v_\mu)$ *then for any closed terms* $t_1,\ldots,t_\mu$ $\phi(t_1,\ldots,t_n) \in T^C|p|$

The next theorem (together with example 1.14) shows that the introduction of "weak" formulas is necessary while dealing with infinitary logics even in they are not with equality whenever we want to have at disposal either $PC_\infty 1$ or E_∞.

THEOREM 1.23. *The following are equivalent:*

(a) *(*) ;*

(b) *for each condition* p $p\|\!\!-\sim\sim\phi$ *if and only if* $p\|\!\!-\phi^{wk}$;

(c) *for each condition* p $p\|\!\!-\sim\sim PC_\infty 1$

and (d) *for each condition* p *from* $p\|\!\!-\sim\sim(\psi\to\phi)$ *for all* $\phi\in\Phi$ *follows* $p\|\!\!-\sim\sim(\psi\to\Lambda\Phi)$.

P r o o f. (a) → (d) Since weak forcing preserves modus ponens (this assertion is a part of Theorem 1.6) (d) is according to Lemma 1.7 equivalent to

(d´) for each condition p $p\|\!\!-\sim\sim(\Lambda_{\phi\in\Phi}\sim\sim(\psi\to\phi)\to(\psi\to\Lambda\Phi))$

i.e. $p\|\!\!-\sim(\Lambda_{\phi\in\Phi}\sim(\psi\wedge\sim\phi)\ \&\ \psi\ \&\sim\Lambda\Phi)$

We think it is simpler to prove (a) → (d´).

Let us suppose (a) holds but for some condition p there exists a condition $q\geq p$ such that $q\|\!\!-\Lambda_{\phi\in\Phi}\sim(\psi\wedge\sim\phi)\ \&\ \psi\ \&\sim\Lambda\Phi$. By 1.12 $q\|\!\!-\sim\Lambda\sim\sim\Phi$ whence for some $\phi_o\in\Phi$ $q\ \|\!\!\not-\sim\sim\phi_o$. Hence for some condition $r\geq q$ $r\|\!\!-\sim\phi_o$ but then $r\|\!\!-\sim(\psi\ \&\sim\phi_o)$ and $r\|\!\!-\psi\ \&\sim\phi_o$, a contradiction.

(d) → (c) By 1.5 (c) from $p\|\!\!-\sim\sim(\Lambda\sim\sim\Phi\to\sim\sim\phi)$ (for any $\phi\in\Phi$) (Lemma 1.10) and $p\|\!\!-\sim\sim(\sim\sim\phi\to\phi)$ follows $p\|\!\!-\sim\sim((\Lambda\sim\sim\Phi\to\sim\sim\phi)\ \&\ (\sim\sim\phi\to\phi))$ thus also $p\|\!\!-\sim\sim(\Lambda\sim\sim\Phi\to\phi)$. Because of (d)

$p\|\!\!-\sim\sim(\Lambda\sim\sim\Phi \to \Lambda\Phi)$ that is $p\|\!\!-\sim(\Lambda\sim\sim\Phi \,\&\, \sim\Lambda\Phi)$.

(c) → (b) On the assumption (c) holds we prove (b) by induction on the complexity of formulas. Of course (see 1.17) the only interesting case is when ϕ is of the form $\Lambda\Psi$

$p\|\!\!-(\Lambda\Psi)^{wk}$ iff for each $\psi\in\Psi$ $p\|\!\!-\psi^{wk}$ iff (by the inductive hypothesis) for each $\psi\in\Psi$ $p\|\!\!-\sim\sim\psi$ iff $p\|\!\!-\Lambda\sim\sim\Psi$ iff $p\|\!\!-\sim\sim\Lambda\Psi$ (in the last step we use: $p\|\!\!-\Lambda\sim\sim\Psi$ and (c) imply that for any $q\geq p$ $q\|\!\!\not-\sim\Lambda\Psi$, consequently $p\|\!\!-\sim\sim\Lambda\Psi$).

(b) → (a) We have just proved that always $p\|\!\!-(\Lambda\Phi)^{wk}$ if and only if $p\|\!\!-\Lambda\sim\sim\Phi$ and (b) gives us also $p\|\!\!-(\Lambda\Phi)^{wk}$ if and only if $p\|\!\!-\sim\sim\Lambda\Phi$.

After all this it seems natural to put the question of the appropriatness of defining a forcing relation taking for the basic symbol Λ rather than V. For with (see |2|) $p\|\!\!-V\Phi$ if and only if for some $\phi\in\Phi$ $p\|\!\!-\phi$ we would obtain the desired $p\|\!\!-\Lambda\Phi$ if and only if $p\|\!\!-\Lambda\sim\sim\Phi$ (where now $\Lambda\Phi$ replaces $\sim V\sim\Phi$) and therefore also

$$p\|\!\!-\phi^{wk} \text{ if and only if } p\|\!\!-\sim\sim\phi$$

which would make the introduction of "weak" formulas unnecessary. Our justification could be that the presented system of axioms and rules of inferences for infinitary logic is in wide use.

Let us here also note that disregarding the way we have defined the forcing relation in case L is a fragment of the language $L_{k^+\omega}$ of power $\leq k$, where k is a regular cardinal and D (a set of new constants) of cardinality κ a condition on C like that we use in Lemma 1.11 is put in order that the Generic Model Theorem holds (|2|).

§ 2. The following example is taken from |1|

EXAMPLE 2.1. Let L be a logic with the set of axioms and rules of inference Λ_o, T a theory consistent in L, A_1 and A_2 sets, respectively, of new constants that is, new function and relation symbols, where $|A_2| \leq |A_1| = \bigcup_{\phi\in L} |\phi| \leq |L|$ $(=\kappa)$ and

$F \subseteq L(A_1 \cup A_2)$ a set with the following properties:

(1) $\phi \in F$ implies $sub(\phi) \subseteq F$ ($sub(\phi)$ is the set of all subformulas of ϕ);

(2) $\phi \in F$ implies $\sim\phi \in F$;

(3) if $\Phi \subseteq F$ and $|\Phi| < \kappa_o$, where κ_o is the supremum of the length of proofs in L (thus λ if L is a fragment of logic $L_{\lambda\mu}$ - we recall the definition: $T \vdash \phi$ iff there is a subset Δ of T such that $|\Delta| < \kappa_o$ and $\vdash \Lambda\Delta \rightarrow \phi$) then $\Lambda\Phi \in F$;

(4) if $\phi(v) \in F$ and $c \in A_1$ then $\phi(c) \in F$;

and (5) for each formula ϕ from F there exists a constant $c \in A_1$ which does not appear in ϕ.

Let C be a set partially ordered by inclusion whose elements are all subsets (p) of $SENT(F) = SENT(L(A_1 \cup A_2)) \cap F$ which satisfy:

(i) if $p \in C$ then $|p| < \kappa_o$

(ii) for $p \in C$ $T \cup p$ is consistent theory in $L(A_1 \cup A_2)$

and (iii) formula belonging to $p \in C$ is not a conjunction.

We define a unary relation $\Vdash$ on $C \times AT(L(A_1 \cup A_2))$ by $p \Vdash \phi$ if and only if $\phi \in p$ and assume that it is extended to a relation on $C \times SENT(L(A_1 \cup A_2))$ resembling the forcing relation.

Our first remark would be that without additional assumptions about the set F the given relation does not have to be a forcing relation Even the assertion:

if t_1, t_2 are closed terms occurring in formulas of F and $\phi(v) \in F$ then

(i) $\emptyset \Vdash \sim\sim (t_1 = t_1)$

and (ii) for each p there exists $q \supseteq p$ such that either $p \not\Vdash t_1 = t_2$ or $p \not\Vdash \phi(t_1)$ or $q \Vdash \phi(t_2)$

does not have to hold always.

It seems most natural to introduce the condition that F contains the complete corresponding finitary logic and then if λ is a singular cardinal necessarily to weaken the condition (3) in order that (5) be kept. The condition (3) is too strong anyhow. Let us also say that the sets $F\Phi_1$, $F\Phi_2$ and F_0 we use in the proof of the weak interpolation theorem do not necessarily satisfy it. In addition to all that (3) is, together with (4), without an extra, great restriction, in collision with (5) for λ singular.

The next results are interesting in themselves and even if a forcing relation is not considered they will be useful in the application of forcing.

Let λ be a regular cardinal (this condition is only to simplify the "story"), L a fragment of Logic $L_{\lambda\mu}$ (with the set of axioms and rules of inference Λ_0) and A_1, A_2 and T is in 2.1 and as for C we omit condition (iii), which in our opinion, in the given consideration does not play any special role. We define the relation $\Vdash$ on $C \times SENT(L(A_1 \cup A_2))$ as in 2.1 with the exception that now we put

$p \Vdash \exists v\phi(v)$ if and only if there exists $c \in A_1$ such that $p \Vdash \phi(c)$ (the alternative would be the strengthening of (4) by

(4′) if $\phi(v) \in F$ and t is a closed term then $\phi(t) \in F$)

For relation $\Vdash$ holds

LEMMA 2.2. $p \Vdash \phi^{wk}$ *if and only if* $p \Vdash \sim\sim \phi^{wk}$.

P r o o f. By induction on the complexity of formulas.

LEMMA 2.3. *If* $\phi \in SENT(F)$, $p \in C$ *and (1)* $\phi \in P$; *(2)* $p \Vdash \phi^{wk}$; *(3)* $T \cup p \nVdash \sim\phi$ *and (4) there exists* q *such that* $q \supseteq p \cup \{\phi\}$ *then*

(a) (1) → (2); (b) (2) → (3) *and* (c) (3) → (4).

P r o o f. (c) is trivial. We prove (a) and (b) (simultaneously) by induction on the complexity of formulas. In regard to the reformulation made we shall give only the supplement (with necessary correction) of the proof from (1).

Let $\wedge\Phi \in p$, $q \supseteq p$ and $\phi \in \Phi$. In view of the fact that $T \cup q \cup \{\phi\}$ is consistent $r = q \cup \{\phi\} \in C$ and by the inductive assumption $r \Vdash \phi^{wk}$. It follows $p \Vdash \sim\sim \phi^{wk}$ i.e. $p \Vdash \phi^{wk}$, accordingly also $p \| \!\vdash (\wedge\phi)^{wk}$.

If $p \| \!- (\wedge\Phi)^{wk}$ but not $T \cup p \| \!\not- \sim \wedge \Phi$ then $T \cup p \cup \{\sim\wedge\Phi\}$ is consistent. Consequently for some $\phi \in \Phi$ $T \cup p \cup \{\sim\wedge\Phi, \sim\phi\}$ is consistent. But for $q \supseteq p \cup \{\sim\wedge\Phi, \sim\phi\}$ $q \| \!\vdash \sim\phi^{wk}$ is in contradiction with $p \| \!- \phi^{wk}$ and $q \| \!\not- \sim\phi^{wk}$ implies the existence of some $r \supseteq q$ such that $r \| \!- \phi^{wk}$ and because of it $T \cup r \not\vdash \sim\phi$ while $\sim\phi \in r$ is a contradiction again.

If $\exists v\phi(v) \in p$ then for some $c \in A_1$ $T \cup p \cup \{\phi(c)\}$ is consistent (it is easy to see that this is true for any constant c of A_1 not appearing in sentences of p). Analogously for any $q \supseteq p$ there exists c from A_1 such that $r = q \cup \{\phi(c)\} \in C$. Since by the inductive hypothesis $r \| \!- \phi^{wk}(c)$ (that is $r \| \!\vdash \exists v\, \phi^{wk}(v)$) $p \| \!- \sim\sim \exists v\phi^{wk}$ i.e. $p \| \!\vdash (\exists v\phi)^{wk}$.

From $p \| \!\vdash (\exists v\phi(v))^{wk}$ follows the existence of $r \in C$ and $c \in A_1$ such that $r \supseteq p$ and $r \| \!\vdash \phi^{wk}(c)$. Assumption $T \cup r \not\vdash \sim\phi(c)$ implies $T \cup r \not\vdash \forall v \sim\phi(v)$, therefore $T \cup p \not\vdash \sim\exists v\phi(v)$.

COROLLARY 2.4. *For* $\phi \in SENT(F)$ *and* $p \in C$

$$p \| \!\vdash \phi^{wk} \quad \textit{if and only if} \quad T \cup p \vdash \phi .$$

REMARK. In regard to the supposition that λ is a regular cardinal the restriction $|T| < \lambda$ is unnecessary. In the opposite case (when λ is singular), we introduce it because of the application of rule E_∞.

The rest of the paper shall be devoted to the proof of the so-called weak interpolation theorem which is to replace, in general, the invalid interpolation theorem for infinitary logics (see |1|, |4|).

THEOREM 2.5. *Let* ϕ_1 *and* ϕ_2 *be two sentences of the given logic* $L_{\lambda\mu}$ *with equality (and the set of axioms and rules of inference* Λ_o*) and let* $\vdash \phi_1 \to \phi_2$ *. Then there exists a sentence* ϕ *of logic* $L_{\lambda\mu}$ *such that* $\vdash \phi_1 \to \phi$ *and* $\vdash \phi \to \phi_2$ *and that each constant and each function and relation symbol with the exception of =, which occurs in* ϕ*, occurs as well in both* ϕ_1 *and* ϕ_2.

The main idea of the proof, which to a great extent suggests technical solutions, is taken on from the proof of the Interpolation Theorem in (classical) $L_{\omega\omega}$ logic where under the assumption that there is no interpolant between ϕ_1 and ϕ_2 is being shown the existence of a Hintikka theory containing ϕ_1 and $\sim\phi_2$, hence the existence of a model (corresponding canonical model) for ϕ_1 and $\sim\phi_2$.

Now in accordance with replacing $\models$ with $\vdash$ we use syntactical apparatus. For that purpose we shall extend, firstly, the language L by a set of (new) constants A of cardinality $\bigcup_{\xi\in L}|\xi|$ and then in $L(A)_{\lambda\mu}$ define set $F\phi_i$, i=1,2 as the set of all formulas with the property that each constant and each function and relation symbol (different from =) of the language L, which occurs in them occurs in ϕ_i and which contain no more constants from A than it is permissible to have quantifiers (which should enable us to "eliminate", if necessary, these constants from the relevant formula). In general $F\phi_i$ satisfies all but the third item of the definition from 2.1. However if $\Phi \subsetneq F\phi_i$ and $|\Phi| < cf\mu$ then $\wedge\Phi \in F\phi_i$. Thus, and by analogy with the proof of Craig's theorem, we are taking for elements of C all the subsets, including the empty set, $P = P_1 \cup P_2$ of $SENT(F\phi_1 \cup F\phi_2)$ (we assume that $P_i \subset F\phi_i$, i=1,2) of the cardinality less than $cf\mu$ and such that the union of theories $Thm(\wedge P_i) = \{\xi \in F_o = F\phi_1 \cap F\phi_2 | \wedge P_i \vdash \xi\}$, i=1,2 is consistent. The relation $|| \subseteq C \times SENT(L(A)_{\lambda\mu})$ is defined like the relation $||\!-$ in 2.3. This time using the sign $||$ instead of $||\!\vdash$ we emphasize that $||$ is not necessarily a forcing relation.

To the proofs of the following several lemmas which can be found in |1| should be added, because it follows from the reasons unmentioned there, the real possibility of the assumption that sets Δ, applied in them, belong to F_o.

LEMMA 2.6. *If* $\xi_i \in F\phi_i$, i=1,2 *and* $\xi \in F_o$ *then* $Thm(\xi_1 \,\&\, \xi) \cup Thm(\xi_2)$ *is consistent in and only if* $Thm(\xi_1) \cup Thm(\xi_2 \,\&\, \xi)$ *is consistent.*

LEMMA 2.7. *If* $\xi_i \in F\phi_i$, $i=1,2$ *and* $Thm(\xi_1) \cup Thm(\xi_2)$ *is inconsistent there exists an interpolant* ξ_o *between* ξ_1 *and* $\sim\xi_2$ ($\vdash \xi_1 \to \xi_o$ *and* $\vdash \xi_o \to \sim\xi_2$).

LEMMA 2.8. *If* $\xi_i \in F\phi_i$, $i=1,2$ *and* $Thm(\xi_1) \cup Thm(\xi_2)$ *is consistent then* $\xi_1 \cup Thm(\xi_1) \cup Thm(\xi_2)$ *is consistent too* .

LEMMA 2.9. *If* $\xi_i \in F\phi_i$, $i=1,2$, $\phi \in F\phi_1$ *and* $Thm(\xi_1 \& \phi) \cup$ $\cup\ Thm(\xi_2)$ *is inconsistent then* $\{\xi_1\} \cup Thm(\xi_2) \vdash \sim\phi$ *and* $Thm(\xi_1 \& \& \sim\phi) \cup Thm(\xi_2)$ *is consistent.*

LEMMA 2.10. *If* $\xi_i \in F\phi_i$, $i=1,2$, $\Lambda\Phi \in F\phi_1$ *and* $Thm(\xi_1 \& \sim\Lambda\Phi) \cup Thm(\xi_2)$ *is a consistent theory then for some* $\phi \in \Phi$ $Thm(\xi_1 \& \sim\Lambda\Phi \& \sim\phi) \cup Thm(\xi_2)$ *is also consistent.*

LEMMA 2.11. *Let* $P \in C$ *and* $\psi \in F\phi_i$ $(i \in \{1,2\})$. *Then* (1) $\to$ (2) *and* (2) $\to$ (3) *where* (1) $\psi \in P$; (2) $P \| \psi^{wk}$ *and* (3) *there exists* Q *in* C *such that* $P \cup \{\psi\} \subseteq Q$.

P r o o f. By induction on the complexity of the formula ϕ. Since lemmas 2.6 - 2.10 have already been given there is little more left to be done. The case ψ is $\exists v\phi(v)$ shall serve as an example (the other cases, we think, are easier).

Let $\exists v\phi(v) \in F\phi_1$ and $\exists v\phi(v) \in P_1 \subseteq P_1 \cup P_2 = P$ (this is, of course, no restriction at all). Then for some $c \in A$ $Thm(\Lambda P_1 \& \& \phi(c)) \cup Thm(\Lambda P_2)$ is consistent for in the opposite case ΛP_1 $\cup Thm(\Lambda P_2) \vdash \sim\phi(c)$ for each $c \in A$ (2.9) but if c is a constant not occurring in either ΛP_1 or ΛP_2 we would obtain $\Lambda P_1 \cup Thm(\Lambda P_2)$ $\vdash \sim \exists v\phi(v)$, contradictory to 2.8 (2.6 makes consideration of theory $Thm(\Lambda P_1) \cup Thm(\Lambda P_2 \& \phi(c))$ in case $\exists v\phi(v) \in F_o$ superfluous. Therefore for all $Q \in C$, $Q \supseteq P$ there exists c in A so that $Q \cup \{\phi(c)\} = R \in C$ whence because of $R \| \phi^{wk}(c)$ $P \| \sim\sim \exists v\phi^{wk}(v)$, accordingly $P \| (\exists v\phi(v))^{wk}$.

On condition $P \| (\exists v\phi(v))^{wk}$ $Q \| \phi^{wk}(c)$ for some $Q \supseteq P$ and some $c \in A$. Then $Q \cup \{\psi(c)\} \subseteq R \in C$ and so $R \cup \{\exists v\phi(v)\} \in C$ also.

However, what else we need is a forcing relation ($\|\!-$) such that a set of conditions is C and for $\phi \in F\phi_i$, $(i \in \{1,2\})$ (and $P \in C$) $P \|\!- \phi^{wk}$ if (and only if) $P \| \phi^{wk}$.

For with it here is the proof. Namely, according to 2.7 it is sufficient to prove $\{\phi_1,\sim\phi_2\} \notin C$. But the assumption $P = \{\phi_1,\sim\phi_2\} \in C$ leads to a contradiction because of theorem 1.22 (and hypothesis $\vdash \phi_1 \to \phi_2$) $P \| \!\!- \sim(\phi_1 \& \sim\phi_2)^{wk}$ while according to 2.11 $P \| \phi_1^{wk} \& \sim\phi_2^{wk}$, thus also $P \| \!\!- \phi_1^{wk} \& \sim\phi_2^{wk}$.

In the following we assume that language L contains of the nonlogical symbols only those occurring in formulas ϕ_1 and ϕ_2 and if it is not already included, the relation symbol $=$, that is $L = L_1 \cup L_2 \cup \{=\}$ where L_1 and L_2 are languages, elements of which are symbols from ϕ_1 and ϕ_2, respectively.

For a closed term t of the language L(A) we will say that it is basic if either t is a constant or t is of the form $f(c_1,\ldots,c_n)$ where f is an n-ary function symbol and $c_1,\ldots,c_n$ are elements of A |5|.

LEMMA 2.12. *For each basic term* t *and each* $P \in C$ *there exists* $c \in A$ *and* $Q \in C$ *such that* $P \cup \{t=c\} \subseteq Q$.

P r o o f. An immediate consequence of lemmas 2.8, 2.9.

Let t be a closed term, $P \in C$ and $c \in A$. We define relation $P \| \!\!- t = c$ in the following way:

for $t = c \in F\phi_1 \cup F\phi_2$ by : (a) $P \| \!\!- t = c$ if and only of $t = c \in P$;

otherwise, inductively (on the complexity of the term t) and according to (a) :if $t = f(t_1,\ldots,t_n)$ (and $f(t_1,\ldots,t_n) = c \notin F\phi_1 \cup F\phi_2$) $P \| \!\!- t = c$ if and only if there exist elements $c_1,\ldots,c_n$ from A such that $P \| \!\!- t_i = c_i$, $i=1,\ldots,n$ and $P \| \!\!- f(c_1,\ldots,c_n) = c$.

Relation $P \| \!\!- c = t$ is analogously determined.

From now on we shall not always accent that the first components (P,Q,R,...) of relation $\| \!\!-$ are elements of C (i.e. we shall not permanently repeat $P \in C$, $Q \in C$, $R \in C$,...). Besides that being most of the proofs of the subsequent lemmas rather tedious than difficult we shall usually omit them.

LEMMA 2.13. *If* $P \Vdash t = c$ $(P \Vdash c = t)$ *and* $Q \supseteq P$ *then* $Q \Vdash t = c$ $(Q \Vdash c = t)$.

LEMMA 2.14. *If* $P \Vdash t = c$ $(P \Vdash c = t)$ *there exists* $Q, Q \supseteq P$ *and* $Q \Vdash c = t$ $(Q \Vdash t = c)$.

LEMMA 2.15. *For each closed term* t *and any* P *there exist* Q *and* $c \in A$ *such that* $P \subseteq Q$ *and* $Q \Vdash t = c$ $(Q \Vdash c = t)$.

P r o o f. If all the symbols of t are from $L_1 (L_2)$ the assertion follows from the fact that there exists a constant $c \in A$ not occurring in the sentences of P. Otherwise we use the induction on the complexity of t.

LEMMA 2.16. *Let* t_1, t_2 *be closed terms and* $c, d \in A$.

(a) *If* $P \Vdash t_1 = c$ *and* $P \Vdash t_1 = d$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q \Vdash c = d$.

(b) *If* $P \Vdash t_1 = c$ *and* $P \Vdash c = d$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q \Vdash t_1 = d$

(c) *If* $P \Vdash t_1 = t_2$ *and* $P \Vdash t_1 = c$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q \Vdash t_2 = c$.

(d) *If* $P \Vdash t_1 = t_2$, $P \Vdash t_1 = c$ *and* $P \Vdash t_2 = d$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q \Vdash c = d$.

Now in the natural way we extend the relation $\Vdash$ with the remark that we shall use the same symbol for extensions.

DEFINITION 2.17. *Let* $\phi \equiv \rho(t_1, \ldots, t_n)$ *(for* $n = 2$ ρ *can be* $=$ *as well) be an atomic sentence (of the language* $L(A)$*) and* $P \in C$. *Relation* $\Vdash \subseteq C \times AT(L(A))$ *is given by*

(a) $P \Vdash \phi$ *if and only if* $\phi \in P$ *for* $\phi \in F\phi_1 \cup F\phi_2$, *that is*

(b) $P \Vdash \phi$ *if and only if there exist constants* $c_1, \ldots, c_n$ *from* A *such that* $P \Vdash t_i = c_i$, $i = 1, \ldots, n$ *and* $\rho(c_1, \ldots, c_n) \in P$ *(i.e. according to (a)* $P \Vdash \rho(c_1, \ldots, c_n)$*) if* $\phi \notin F\phi_1 \cup F\phi_2$

(Clearly, requirements $P \Vdash c_i = t_i$ *instead of* $P \Vdash t_i = c_i$, $i = 1, \ldots, n$ *would change nothing essentially – 2.14).*

LEMMA 2.18. *If* ϕ *is an atomic sentence,* $P \Vdash \phi$ *and* $Q \supseteq P$ *then also* $Q \Vdash \phi$.

LEMMA 2.19. *For all closed terms* t_1, t_2, t_3 *and each* $P \in C$ *holds:*

(a) *There exists* $Q, Q \supseteq P$ *and* $Q \Vdash t_1 = t_1$;

(b) *if* $P \Vdash t_1 = t_2$ *there exists* $Q, Q \supseteq P$ *and* $Q \Vdash t_2 = t_1$;

(c) *if* $P \Vdash t_1 = t_2$ *and* $P \Vdash t_2 = t_3$ *there exists* $Q, Q \supseteq P$ *and* $Q \Vdash t_1 = t_3$.

LEMMA 2.20. *Let* $t_i = t_i'$, $i=1,\dots,n$ *be closed terms,* f *and* ρ *a function that is a relation symbol of the length* n *and* $P \in C$. *If* $P \Vdash t_i = t_i'$, $i=1,\dots,n$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q \Vdash f(t_1,\dots,t_n) = f(t_1',\dots,t_n')$. *If still* $P \Vdash \rho(t_1,\dots,t_n)$ *there exists* $R, R \supseteq P$ *and* $R \Vdash \rho(t_1',\dots,t_n')$.

LEMMA 2.21. *Let* t_1, t_2 *and* σ *be closed terms,* σ' *a term obtained by substitution in* σ *(not necessarily all) occurences of* t_1 *by* t_2 *and let* $P \in C$. *If* $P \Vdash t_1 = t_2$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q \Vdash \sigma = \sigma'$.

P r o o f. By induction on the complexity of σ using the previous lemmas.

LEMMA 2.22. *Let* t_1, t_2 *be closed terms and* $\phi(v)$ *an atomic formula (with at most one free variable). Then for each* P *there exists* $Q \supseteq P$ *such that either* $P \not\Vdash t_1 = t_2$ *or* $P \not\Vdash \phi(t_1)$ *or* $Q \Vdash \phi(t_2)$.

According to 2.18 and 2.22 the relation $\Vdash$ determines a forcing relation (which we denote also by $\Vdash$ and) for which holds:

LEMMA 2.23. *Let* P *be a condition (now in regard to the accepted terminology we again call elements of* C *conditions),* t *a closed term and* $\phi(v)$ *a formula (of the language* L(A)) *with at most one free variable,* c *an element of* A *and let* $P \Vdash t = c$. *Then* $P \Vdash (\phi(t))^{wk}$ *if and only if* $P \Vdash (\phi(c))^{wk}$.

P r o o f. By induction on the complexity of formula ϕ. If $\phi(v)$ is an atomic formula the statement is a direct consequence of the previous three lemmas. Other cases are trivial.

On the basis of what has been said the proof of (the Weak Interpolation Theorem) 2.5 follows from

THEOREM 2.24. *For* $\phi \in F\phi_1 \cup F\phi_2$ *and* $P \in C$
$P \| \phi^{wk}$ *if and only if* $P \| \!\vdash \phi^{wk}$.

P r o o f. By induction on the complexity of ϕ. The only interesting case we have is when ϕ is of the form $\exists v\psi(v)$ (where $\psi(v)$ is a formula with at most one free variable).

Let $P \| \!\vdash (\exists v\psi(v))^{wk}$ (i.e. $P \| \!\vdash \sim\sim \exists v\psi^{wk}(v)$) and $Q \supseteq P$. Then for some $R \supseteq Q$ and some closed term t $R \| \!\vdash \psi^{wk}(t)$. By 2.15 there exists $S(\in C)$ and $c \in A$ such that $S \supseteq R$ and $S \| \!\vdash t = c$. Thus also (2.23) $S \| \!\vdash \psi^{wk}(c)$ whence $P \| (\exists v\psi(v))^{wk}$ too. The proof in the opposite direction is trivial.

REFERENCES

|1| Jean-Pierre Keller, *Abstract Forcing and Applications PhD. Thesis, New York University (1977)*

|2| H.J.Keisler, *Forcing and the Omitting Types Theorem, Studies in Mathematics (ed. A.Morley), Vol. 8, 96-133.*

|3| K.J.Barwise, A.Robinson, *Completing Theories by Forcing, Annals of Mathematical Logic, Vol. 2 (1970), 119-142.*

|4| M.A.Dickmann, *Large Infinitary Languages, North-Holland, (Amsterdam), (1975).*

|5| R.Gostanian, *Lectures on Model Theory, part I, Lecture Notes Series, (1974).*

Received by the editors June 20,1983.

REZIME

PRIMEDBA O FORSINGU I ŠLABOJ INTERPOLACIONOJ TEOREMI ZA BESKONAČNE LOGIKE

Naša razmatranja odnose se na rezultate prva tri poglavlja iz |1|. Cilj nam je da bolje osetimo ulogu slabih formula i korigujemo dokaz slabe interpolacione teoreme za beskonačne logike.

U |1| je već nagovešteno da (direktni) dokazi tvrdjenja kao što su: za svako p $p \| \vdash \sim\sim PC_{\infty}1$, ili: pravilo izvodjenja E_{∞} ostaje očuvano, "izgleda da zahtevaju" sledeće svojstvo skupa uslova C: (*) za svako p iz $p \| \vdash \Lambda \sim\sim \Phi$ sledi $p \| - \sim\sim \Lambda \Phi$ (implikacija u suprotnom smeru je uvek tačna). Mi pokazujemo da su ovi iskazi ekvivalentni (dakle i medjusobno ekvivalentni) sa: za svako p $p \| - \sim\sim\phi$ ako i samo ako $p \| \vdash \phi^{wk}$ (Teorema 1.23). Dovoljan uslov, samo pretpostavljen u |1|, da ova tvrdjenja i važe u slučaju kada posmatramo fragment logike $L_{k\lambda}$, je da svaki nerastući niz uslova dužine $\lambda < k$ ima gornje ograničenje (Lema 1.11). To, medjutim, nije i potreban uslov (Primer 1.13). Iz (*) takodje proizilazi: za svako p $p \| \vdash \sim\sim PC11$, što inače nije u opštem ispunjeno za beskonačne logike (Primer 1.15).

Što se tiče dokaza slabe interpolacione teoreme za beskonačne logike (semantičko $\models$ je zamenjeno sintaktičkim $\vdash$) osnovna zamerka nam je da relacija ($\|$) koja se u njemu koristi ne mora da bude i u slučajevima od stvarnog interesa i nije, forsing relacija dok se u isto vreme koriste osobine forsing relacije (posebno 1.22). Iskrsli problem rešavamo konstrukcijom forsing relacije koja sa datom ima presek "po meri" (Leme 2.12 - 2.23, Teorema 2.24). Učinjene su i neke druge korekcije i poboljšanja.

Komentari uz pojedine stavove treba da doprinesu boljem sagledavanju izložene materije.