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# A GENERALIZATION OF THE BOCHNER AND CONSTANT BOCHNER CURVATURE TENSOR

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ABSTRACT

Using the curvature tensor of the complex conformal connection, the tensor (2.28) is constructed. This tensor generalizes the Bochner curvature tensor as well as the complex conharmonic curvature tensor.

Using the curvature tensor of the contact conformal connection, the tensor (5.6) is constructed. This tensor generalizes the contact Bochner curvature tensor as well as the contact conharmonic curvature tensor.

## INTRODUCTION

Let $(M,g,F)$ be the Kähler manifold, i.e. an n-dimensional ($n = 2m$; $n \geq 4$) differentiable manifold of class $C^\infty$ covered by a system of coordinate neighborhoods $(U,x^i)$ in which there are given a tensor field $F_j^i$ and a Riemannian metric $g_{ij}$ satisfying

$$F_j^i F_i^k = -\delta_j^k, \quad g_{ab}F_i^a F_j^b = g_{ij}, \qquad \nabla_j F_i^k = 0,$$

where $\nabla_j$ is the operator of the covariant differentiation with respect to the Christoffel symbols $\{ {k \atop ji} \}$ formed with $g_{ij}$.

If we put

$$F_{ji} = F_j^a g_{ai} ,$$

we have

$$F_{ij} = - F_{ji}, \qquad \nabla_k F_{ij} = 0.$$

Trying to find a complex analogue of the conformal change of the Riemannian connection, K. Yano |1| considered a conformal change of the Hermitian metric:

(1.1) $\bar{g}_{ij} = e^{2\sigma} g_{ij}, \quad \bar{F}_j^i = F_j^i, \quad \bar{F}_{ij} = e^{2\sigma} F_{ij}$

where $\sigma$ is a scalar function, and he looked for an affine connection $\Gamma_{jk}^i$ such that

$$D_k \bar{g}_{ij} = 0,$$

where $D_k$ denotes the operator of the covariant differentation with respect to $\Gamma_{jk}^i$, and such that the torsion tensor $S_{ji}^k$ is given by

$$S_{ji}^k = - F_{ji}\kappa^k ,$$

where $\kappa^k$ are the components of a vector field. This complex conformal connection is given by

(1.2) $$\Gamma_{ji}^k = \{ {k \atop ji} \} + \delta_j^k \sigma_i + \delta_i^k \sigma_j - g_{ji}\sigma^k + F_j^k \kappa_i + F_i^k \kappa_j - F_{ji}\kappa^k ,$$

were

(1.3) $$\sigma_i = \partial_i \sigma, \quad \sigma^i = \sigma_a g^{ai}, \quad \kappa_i = -\sigma_a F_i^a, \quad \kappa^k = \kappa_a g^{ak} .$$

In the same paper K.Yano proved the Theorem:

If, in a real n-dimensional Kähler manifold $(n \geq 4)$, there exists a scalar function $\sigma$ such that the complex conformal connection (1.2) is of zero curvature, then the Bochner curvature tensor:

(1.4.) $$\begin{aligned} B_{kji}{}^h = K_{kji}{}^h + \frac{1}{n+4} (\delta_j^h K_{ki} - \delta_k^h K_{ji} - g_{ji} K_k{}^h + g_{ki} K_j{}^h \\ + F_j^h K^*_{ki} - F_k^h K^*{}_{ji} - F_{ji} K^*{}_k{}^h + F_{ki} K^{*h}_j + 2K^*_{kj} F_i^h + \\ + 2K^*{}_i{}^h F_{kj}) - \frac{K}{(n+2)(n+4)} (\delta_j^h g_{ki} - \delta_k^h g_{ji} + \\ + F_j^h F_{ki} - F_k^h F_{ji} + 2F_{kj} F_i^h) \end{aligned}$$

where $K_{kji}{}^h$ is the curvature tensor of $\{ {k \atop ji} \}$ and

$$K_{ji} = K_{aji}{}^h, \quad K = K_{ij} g^{ij}, \quad K^*{}_{ji} = - K_{ja} F_i^a, \quad K^*{}_j{}^h = K^*{}_j{}^h g^{ah}$$

In paper |2| ,K.Yano found the contact analogue of the above.

Namely, let $(M,\phi,\xi,\eta,g)$ be the Sasakian manifold, i.e. a $(2m+1)$-dimensional differentiable manifold of class $C^\infty$ covered by a system of coordinate neighborhoods $(U,x^i)$ in which there are given a tensor field $\phi_i^h$, a vector field $\xi^h$, a 1-form $\eta_i$ and a Riemannian metric $g_{ij}$ satisfying

$$\phi_j^i\phi_i^h = -\delta_j^h + \eta_j\xi^h, \quad \phi_i^h\xi^i = 0, \; \eta_i\phi_j^i = 0, \quad \eta_i\xi^i = 1,$$

$$g_{ab}\phi_j^a\phi_i^b = g_{ji} - \eta_j\eta_i, \quad \eta_i = g_{ih}\xi^h,$$

$$\phi_{ji} \equiv \phi_j^a g_{ai} = \frac{1}{2}(\partial_j\eta_i - \partial_i\eta_j)$$

$$\nabla_i\eta^h = \phi_i^h, \quad \nabla_j\phi_i^h = -g_{ji}\eta^h + \delta_j^h\eta_i .$$

Then K.Yano found a contact conformal connection as follows |2|:

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} + (\delta_j^h - \eta_j\eta^h)\sigma_i + (\delta_i^h - \eta_i\eta^h)\sigma_j$$

(1.5)

$$- (g_{ji} - \eta_j\eta_i)\sigma^h + \phi_j^h(\kappa_i - \sigma_i) + \phi_j^h(\kappa_j - \sigma_j) - \phi_{ji}(\kappa^h - \sigma^h) ,$$

where

$$\sigma_i = \partial_i\sigma, \quad \sigma^h = \sigma_a g^{ah}, \quad \kappa_i = -\sigma_a\phi_i^a, \quad \kappa^h = \kappa_a g^{ah} .$$

This connection satisfies

$$\mathcal{D}_k(e^{2\sigma}g_{ij}) = 2e^{2\sigma}\sigma_k\eta_i\eta_j, \quad \mathcal{D}_k\phi_j^i = 0, \quad \mathcal{D}_k\eta^h = 0,$$

i.e.

$$\mathcal{D}_k\left[e^{2\sigma}(g_{ij} - \eta_i\eta_j)\right] = 0.$$

where $\mathcal{D}_k$ is the operator of the covariant differentation with respect to (1.5), and the function $\sigma$ satisfies $\sigma_i\eta^i = 0$.

Also, K.Yano proved the Theorem |2|:

If, in a $(2m+1)$-dimensional Sasakian manifold $(2m+1 > 3)$, there exists a scalar function $\sigma$ such that the contact conformal connection (1.5) is of zero curvature, then the contact Bochner curvature tensor:

$$
\begin{aligned}
B_{kji}{}^{h} = {} & K_{kji}{}^{h} \\
& -(\delta_k^h - \eta_k\eta^h)\left(\frac{K_{ji}}{2(m+2)} + \frac{6m+8-K}{8(m+1)(m+2)}\, g_{ji} + \frac{10m+8+K}{8(m+1)(m+2)}\,\eta_i\eta_j\right) \\
& +(\delta_j^h - \eta_j\eta^h)\left(\frac{K_{ki}}{2(m+2)} + \frac{6m+8-K}{8(m+1)(m+2)}\, g_{ki} + \frac{10m\ +8+K}{8(m+1)(m+2)}\,\eta_i\eta_k\right) \\
& -(g_{ji} - \eta_j\eta_i)\left(\frac{K_k^{\ h}}{2(m+2)} + \frac{6m+8-K}{8(m+1)(m+2)}\,\delta_k^i + \frac{10m+8+K}{8(m+1)(m+2)}\,\eta_k\eta^h\right) \\
(1.6)\qquad & +(g_{ki} - \eta_k\eta_i)\left(\frac{K_j^{\ h}}{2(m+2)} + \frac{6m+8-K}{8(m+1)(m+2)}\,\delta_j^k + \frac{10m+8+K}{8(m+1)(m+2)}\,\eta_j\eta^h\right) \\
& +\phi_k^h\left(\frac{K_{ja}\phi_i^a}{2(m+2)} - \frac{6m+8-K}{8(m+1)(m+2)}\,\phi_{ji}\right) - \phi_j^h\left(\frac{K_{ka}\phi_i^a}{2(m+2)} - \frac{6m+8-K}{8(m+1)(m+2)}\,\phi_{ki}\right) \\
& +\phi_{ji}\left(\frac{K_{ka}\phi^{ha}}{2(m+1)} - \frac{6m+8-K}{8(m+1)(m+2)}\,\phi_k^h\right) - \phi_{ki}\left(\frac{K_{ja}\phi^{ha}}{2(m+2)} - \frac{6m+8-K}{8(m+1)(m+2)}\,\phi_j^h\right) \\
& -\frac{1}{m+2}\left(K_{ia}\phi^{ha}\phi_{kj} + K_{ka}\phi_j^a\phi_i^h\right) + \left(\frac{3m+4}{(m+1)(m+2)} - \frac{K}{2(m+1)(m+2)}\right)\phi_{kj}\phi_i^h \\
& +\phi_k^h\phi_{ji} - \phi_j^h\phi_{ki} - 2\phi_{kj}\phi_i^h
\end{aligned}
$$

vanishes.

In the present paper we shall determine the tensors generalizing the Bochner curvature tensor and the contact Bochner curvature tensor, respectively. Namely, in section 2, we shall consider the Kähler manifold where there exists a scalar function $\sigma$ such that

$$(1.7)\qquad \nabla_p\sigma^p + a\sigma_p\sigma^p = 0, \quad a = \text{const}; \quad a \neq \frac{n^2-4}{2(n+1)} .$$

Then, using the curvature tensor of the complex conformal connection (1.2) we construct the tensor $H_{kji}{}^{h}$ which does not depend on the function $\sigma$. In the case $a = n+2$, $H_{kji}{}^{h}$ reduce to the Bochner curvature tensor. In section 3 we shall give some theorems conserning the tensor $H_{kji}{}^{h}$. In section 4 we shall find the complex conharmonic curvature tensor which is another special case of $H_{kji}{}^{h}$.

In section 5 we consider (2m+1)-dimensional Sasakian manifold where there exists a scalar function $\sigma$ satisfying (1.7) such that $a \neq \frac{2(m^2-1)}{2m+1}$. Using the curvature tensor of the contact conformal connection (1.5) we construct the tensor $\mathcal{H}_{kji}{}^{h}$ generalizing the contact Bochner curvature tensor as well as the conharmonic curvature tensor.

## 2. GENERALIZATION OF THE BOCHNER CURVATURE TENSOR

We compute the curvature tensor

$$R_{kji}{}^{h} = \partial_k \Gamma_{ji}^{h} - \partial_j \Gamma_{ki}^{h} + \Gamma_{kt}^{h} \Gamma_{ji}^{t} - \Gamma_{jt}^{h} \Gamma_{ki}^{t}$$

of the complex conformal connection (1.2) and we follow the way indicated in [1]. We find

$$(2.1)\qquad \begin{aligned} R_{kji}{}^{h} &= K_{kji}{}^{h} + \delta_j^h p_{ki} - \delta_k^h p_{ji} - p_k{}^h g_{ji} + p_j{}^h g_{ki} \\ &\quad + F_j^h q_{ki} - F_k^h q_{ji} - q_k{}^h F_{ji} + q_j{}^h F_{ki} - \alpha_{kj} F_i^h - \beta_i{}^h F_{kj} , \end{aligned}$$

where

$$(2.2)\qquad \begin{aligned} p_{ji} &= \nabla_j \sigma_i - \sigma_i \sigma_j + \kappa_i \kappa_j + \frac{1}{2}\sigma_a \sigma^a g_{ij} , \\ q_{ji} &= \nabla_j \kappa_i - \sigma_j \kappa_i - \kappa_j \sigma_i + \frac{1}{2}\sigma_a \sigma^a F_{ji} , \\ p_k{}^h &= p_{ka} g^{ah}, \qquad q_k{}^h = q_{ka} g^{ah} , \end{aligned}$$

$$(2.3)\qquad \alpha_{kj} = -(\nabla_k \kappa_j - \nabla_j \kappa_k), \quad \beta_i{}^h = 2(\sigma_i \kappa^h - \kappa_i \sigma^h) .$$

Consequently

$$(2.4)\qquad q_{ji} = -p_{ja} F_i^a , \quad p_{ji} = q_{ja} F_i{}^a ,$$

$$(2.5)\qquad F^{kj} \alpha_{kj} = 2 \nabla_a \sigma^a , \quad F^{kj} \beta_{kj} = -4 \sigma_a \sigma^a .$$

The vector $\sigma_i$ being the gradient, we have $p_{ij} = p_{ji}$, too. We rewrite (2.1) in the form:

$$
\begin{aligned}
R_{kjih} &= K_{kjih} - g_{kh}p_{ji} + g_{jh}p_{ki} - p_{kh}g_{ji} + p_{jh}g_{ki} \\
&\quad - F_{kh}q_{ji} + F_{jh}\,{}_{ki} - {}_{kh}F_{ji} + g_{jh}F_{ki} \\
&\quad - \alpha_{kj}F_{ih} - \beta_{ih}F_{kj} \,. 
\end{aligned} \tag{2.6}
$$

In (2.6) we interchange the indices k and i as well as j and h and substract from (2.6). Then we have

$$
\begin{aligned}
R_{kjih} - R_{ihkj} &= -F_{kh}(q_{ji}+q_{ij}) + F_{jh}(q_{ki}+q_{ik}) - F_{ji}(q_{kh}-q_{hk}) \\
&\quad + F_{ki}(q_{jh}+q_{hj}) - F_{ih}(\alpha_{kj}-\beta_{kj}) + F_{kj}(\alpha_{ih}-\beta_{ih}) \,.
\end{aligned} \tag{2.7}
$$

Transvecting with $F^{kh}$, we obtain

$$
q_{ji}+q_{ij} = \frac{F^{rs}(R_{rjis} - R_{isrj})}{n+2} \tag{2.8}
$$

Substituting (2.8) into (2.7), we get

$$
-F_{ih}(\alpha_{kj}-\beta_{kj}) + F_{kj}(\alpha_{ih}-\beta_{ih}) = M_{kjih} \,, \tag{2.9}
$$

where

$$
\begin{aligned}
M_{kjih} &= R_{kjih} - R_{ihkj} \\
&\quad + \frac{F^{rs}}{n+2}\Big[(R_{rjis}-R_{isrj})F_{kh} - (R_{rkis}-R_{isrk})F_{jh} \\
&\quad + (R_{rkhs}-R_{hsrk})F_{ji} - (R_{rjhs}-R_{hsrj})F_{ki}\Big] \,.
\end{aligned} \tag{2.10}
$$

Transvecting (2.9) with $F^{kj}$ and using (2.5), we find

$$
\alpha_{ih} - \beta_{ih} = -\frac{2}{n}(\nabla_a\sigma^a + 2\sigma_a\sigma^a)F_{ih} - F^{kj}M_{kjih} \,. \tag{2.11}
$$

On the other hand, we find from (2.2):

$$
q_{ji} - q_{ji} = \nabla_j\kappa_i - \nabla_i\kappa_j + \sigma_a\sigma^a F_{ji} \,.
$$

But (2.8) gives

$$
-\,q_{ij} = q_{ji} - \frac{F^{rs}(R_{rjis} - R_{isrj})}{n-2} \,.
$$

Therefore we have

$$\alpha_{ji} = -(\nabla_j\kappa_i - \nabla_i\kappa_j) = -2q_{ji} + \sigma_a\sigma^a F_{ji} + \frac{F^{rs}(R_{rjis}-R_{isrj})}{n-2} .$$

If we put

$$A_{ji} = \frac{F^{rs}(R_{rjis}-R_{isrj})}{n-2} , \tag{2.12}$$

we rewrite the preceding relation in the form:

$$\alpha_{ji} = -2q_{ji} + \sigma_a\sigma^a F_{ji} + A_{ji} . \tag{2.13}$$

Using (2.13), we obtain from (2.11):

$$\beta_{ji} = -2q_{ji} + \frac{2}{n}(p_a{}^a + 2\sigma_a\sigma^a)F_{ji} + B_{ji} , \tag{2.14}$$

where

$$B_{ji} = \frac{F^{rs}(R_{rjis}-R_{isrj})}{n-2} + F^{kh} M_{khji} . \tag{2.15}$$

Now, in (2.1) we contract with respect to h and k and use the skew symmetry of the tensor $\alpha_{ij}$. Then putting $R_{aji}{}^a = R_{ji}$, we obtain

$$\begin{aligned} R_{ji} &= K_{ji} + (2-n)p_{ji} - p_a^a g_{ji} + \\ &+ F_j^a q_{ai} - q_a^a F_{ji} + q_j{}^a F_{ai} + \alpha_{ja}F_i^a + \beta_{ia}F_j^a . \end{aligned} \tag{2.16}$$

On the other hand, taking into account (2.8), (2.12) and (2.2), we find

$$q_{ai}F_j^a = -p_{ji} + A_{ai}F_j^a .$$

Substituting this into (2.16), we have

$$R_{ji} - A_{ai}F_j^a = K_{ji} - np_{ji} - p_a^a g_{ji} - q_a^a F_{ji} + \alpha_{ja}F_i^a + \beta_{ia}F_j^a . \tag{2.17}$$

Transvecting (2.17) with $F^{ji}$, we get

$$q_a^a = \frac{1}{n}(R_{ji}F^{ji} + A_{ai}g^{ai}) .$$

Substituting this into (2.17), we obtain

$$R_{ji} - A_{ai}F^a_j + \frac{1}{n}(R_{ab}F^{ab} + A_{ab}g^{ab})F_{ji} =$$

(2.18)

$$= K_{ji} - np_{ji} - p_a{}^a g_{ji} + \alpha_{ja}F^a_i + \beta_{ia}F^a_j \quad .$$

But, taking into account (2.13) and (2.14), we find, respectively,

$$\alpha_{ja}F^a_i = -2p_{ji} + \sigma_a\sigma^a g_{ji} + A_{ja}F^a_i \ ,$$

$$\beta_{ia}F^a_j = -2p_{ji} + \frac{2}{n}\ (p_a{}^a + 2\sigma_a\sigma^a)g_{ji} + B_{ia}F^a_j \ .$$

Therefore, (2.18) can be written in the form:

(2.19) $$\bar{R}_{ji} = K_{ji} - (n+4)p_{ji} + (\ \frac{2-n}{n}\ p_a{}^a + \frac{n+4}{n}\ \sigma_a\sigma^a)g_{ij}\ ,$$

where we have put

(2.20) $$\bar{R}_{ji} = R_{ji} + \frac{1}{n}\ (R_{ab}F^{ab} + A_{ab}g^{ab})F_{ji} - (A_{ai}F^a_j + A_{ja}F^a_i) - B_{ia}F^a_j \ .$$

Now, we shall use condition (1.7). Then we have

$$p_a{}^a = \nabla_a\sigma^a + \frac{n}{2}\ \sigma_a\sigma^a = \frac{n-2a}{2}\ \sigma\ {}_a\sigma^a_h\ ,$$

and (2.19) reduce to

(2.21) $$\bar{R}_{ji} = K_{ji} - (n+4)p_{ji} + \frac{1}{2n}\ [(2-n)(n-2a)+2(n+4)]\sigma_a\sigma^a g_{ji}\ ,$$

from which, transvecting with $g^{ij}$, we get

$$e^{2\sigma}\bar{R} = K + [(n+4)-(n+1)(n-2a)]\ \sigma_a\sigma^a, \quad (\bar{R} = \bar{R}_{ji}\bar{g}^{ji})$$

i.e.

$$\sigma_a\sigma^a = \frac{e^{2\sigma}\bar{R} - K}{n+4-(n+1)(n-2a)}$$

Substituting this into (2.21) and putting

(2.22) $$b = (2-n)(n-2a) + 2(n+4), \quad c = n+4-(n+1)(n-2a),$$

we find

(2.23) $$p_{ji} = -\frac{\bar{R}_{ji}}{n+4} + \frac{K_{ji}}{n+4} + \frac{b}{2n(n+4)c}\ (e^{2\sigma}\bar{R} - K)g_{ji}\ .$$

Then, using (2.4) and the fact that, in the Kähler space,

$$-K_{ja}F_i^a = K_{ai}F_j^a = K^*_{ji} ,$$

we obtain

$$(2.24)\qquad q_{ji} = -p_{ja}F_i^a = \frac{\bar{R}_{ja}F_i^a}{n+4} + \frac{K^*_{ji}}{n+4} + \frac{b}{2n(n+4)c}(e^{2\sigma}\bar{R} - K)F_{ji} .$$

Also, taking into account (2.13), we find

$$(2.25)\qquad \alpha_{ji} = -2\frac{\bar{R}_{ja}F_i^a}{n+4} - 2\frac{K^*_{ji}}{n+4} + \frac{1}{nc}\left(n - \frac{b}{n+4}\right)(e^{2\sigma}\bar{R}-K)F_{ji} + A_{ji} .$$

Finally, taking into account (2.14), we have

$$(2.26)\qquad \beta_{ji} = -2\frac{\bar{R}_{ja}F_i^a}{n+4} - 2\frac{K^*_{ji}}{n+4} + \frac{1}{nc}\left(n-2a+4 - \frac{b}{n+4}\right)(e^{2\sigma}\bar{R}-K)F_{ji} + B_{ji} .$$

Substituting (2.23),(2.24),(2.25) and (2.26) into (2,1), we obtain

$$(2.27)\qquad \begin{aligned} & R_{kji}{}^h + \frac{1}{n+4}(\delta_j^h\bar{R}_{ki} - \delta_k^h\bar{R}_{ji} - \bar{g}_{ji}\bar{R}_k{}^h + \bar{g}_{ki}\bar{R}_j{}^h - \\ & - F_j{}^h\bar{R}_{ka}F_i^a + F_k^h\bar{R}_{ja}F_i^a + \bar{F}_{ji}\bar{R}_{ka}\bar{F}^{ah} - \bar{F}_{ki}\bar{R}_{ja}\bar{F}^{ah} - \\ & - 2\bar{R}_{ka}F_j^aF_i^h - 2\bar{R}_{ia}\bar{F}^{ah}\bar{F}_{kj}) + \frac{\bar{R}}{n(n+4)c}\left[-b(\delta_j{}^h\bar{g}_{ki} - \right. \\ & - \delta_k^h\,\bar{g}_{ji} + F_j{}^h\bar{F}_{ki} - F_k{}^h\bar{F}_{ji}) + 2n(2n+2-3a)\bar{F}_{kj}F_i^h\left.\right] + \\ & + A_{kj}F_i{}^h + B_i{}^h\bar{F}_{kj} = K_{kji}{}^h + \frac{1}{n+4}(\delta_j{}^hK_{ki} - \\ & - \delta_k^hK_{ji} - g_{ji}K_k{}^h + g_{ki}K_j{}^h + F_j{}^hK^*_{ki} - F_k{}^hK^*_{ji} - F_{ji}K^{*h}_k + \\ & + F_{ki}K^{*h}_j + 2K^*_{kj}F_i{}^h + 2K^{*h}_iF_{kj}) + \frac{K}{n(n+4)c}\left[-b(\delta_j{}^hg_{ki} - \right. \\ & - \delta_k^hg_{ji} + F_j^hF_{ki} - F_k{}^hF_{ji}) + \\ & + 2n(2n+2-3a)F_{kj}F_i{}^h\left.\right] \end{aligned}$$

The tensor on the right hand side of (2.27), being independent of the function $\sigma$, isso the tensor of the left hand side. Therefore we have.

THEOREM. *If in a real n-dimensional Kähler space* $(n \geq 4)$ *there exists a scalar function* $\sigma$ *such that the condition (1.7) is satisfied, the thensor on the left hand side of (2.27) is invariant with respect to the complex conformal transformation (1.1).*

Let us put

$$
\begin{aligned}
H_{kji}{}^{h} = {} & K_{kji}{}^{h} + \frac{1}{n+4}(\delta_j^h K_{ki} - \delta_k^h K_{ji} - g_{ji}K_k^h + g_{ki}K_j{}^h + \\
(2.28) \qquad & + F_j{}^h K^*{}_{ki} - F_k^h K^*_{ji} - F_{ji}K_k^{*h} + F_{ki}K_j^{*h} + 2K^*_{kj}F_i{}^h + 2K_i^{*h}F_{kj}) \\
& + \frac{K}{n(n+4)c}\left[-b(\delta_j^h g_{ki} - \delta_k^h g_{ji} + F_j{}^h F_{ki} - F_k{}^h F_{ji}) + \right. \\
& \left. + 2n(2n+2-3a)F_{kj}F_i{}^h\right] .
\end{aligned}
$$

This tensor generalizes the Bochner curvature tensor. In fact, if $a = n \neq 2$, then $b = n(n+4)$, $c = (n+2)(n+4)$, $2n(2n+2-3a) = -2n(n+4)$ and (2.28) reduces to the Bochner curvature tensor (1.4).

Moreover, we mention the case $a = \frac{n}{2}$. Then $b = 2(n+4)$, $c = n+4$, $2n(2n+2-3a) = n(n+4)$ and $H_{kji}{}^h$ has the form

$$
\begin{aligned}
H_{kji}{}^{h} = {} & K_{kji}{}^{h} + \frac{1}{n+4}(\delta_j{}^h K_{ki} - \delta_k{}^h K_{ji} - g_{ji}K_k{}^h + g_{ki}K_j{}^h + \\
(2.29) \qquad & + F_j{}^h K^*_{ki} - F_k{}^h K^*_{ji} - F_{ji}K_k^{*h} + F_{ki}K^*{}_j^h + 2K^*_{kj}F_i{}^h + 2K_i^{*h}F_{kj}) + \\
& + \frac{K}{n+4}\left[-2(\delta_j^h g_{ki} - \delta_k^h g_{ji} + F_j^h F_{ki} - F_k^h F_{ji}) + nF_{kj}F_i{}^h\right] .
\end{aligned}
$$

In section 4 we shall find another method to obtain the tensor (2.29).

## 3. SOME PROPERTIES OF THE TENSOR $H_{kji}{}^{h}$

It is easy to see that the tensor $H_{kji}{}^{h}$ and $H_{kjih} = H_{kji}{}^{a} g_{ah}$ satisfy the following conditions:

$$H_{kjih} = -H_{jkih}, \quad H_{kjih} = -H_{kjhi}, \quad H_{kjih} = H_{ihkj},$$

$$H_{aji}{}^{a} = 0, \qquad H_{kja}{}^{h} F_{i}^{a} - H_{kji}{}^{a} F_{a}^{h} = 0,$$

$$H_{kjih} + H_{jikh} + H_{ikjh} = \frac{2K}{nc}(n+2-a)(F_{kj}F_{ih} + F_{ji}F_{kh} + F_{ik}F_{jh}) \tag{3.1}$$

(i.e. the condition

$$H_{kjih} + H_{jikh} \; H_{ikjh} = 0,$$

is satisfied only in the case when $H_{kji}{}^{h}$ is the Bochner curvature tensor or when the Kähler space is a space with zero scalar curvature).

Now, we prove two theorems concerning the tensor $H_{kji}{}^{h}$ when it differs from the Bochner curvature tensor, i.e. when $a \neq n+2$ and $a \neq \frac{n^2-4}{2(n+1)}$.

THEOREM. *The scalar curvature of the Kähler space with the parallel tensor $H_{kji}{}^{h}$ is constant.*

P r o o f. We find from (2.27) that

$$\begin{aligned} \nabla_t H_{kji}{}^{h} = {} & \nabla_t K_{kji}{}^{h} + \frac{1}{n+4}(\delta_j^h \nabla_t K_{ki} - \delta_k^h \nabla_t K_{ji} - g_{ji} \nabla_t K_k^h + g_{hi} \nabla_t K_j^h \\ & + F_j^h \nabla_t K^*_{ki} - F_k^h \nabla_t K^*{}_{ji} - F_{ji} \nabla_t K^*{}_k{}^h + F_{ki} \nabla_t K^*_j{}^h + \\ & + 2F_i{}^h \nabla_t K^*_{kj} + 2F_{kj} \nabla_t K^*_i{}^h) + \frac{\nabla_t K}{n(n+4)c}[-b(\delta_j^h g_{ki} - \\ & - \delta_k^h g_{ji} + F_j^h F_{ki} - F_k^h F_{ji}) + 2n(2n+2-3a) F_{kj} F_i{}^h] . \end{aligned} \tag{3.2}$$

Contracting with respect to t and h and taking into account that

$$\nabla_t K_{kji}{}^{t} = \nabla_k K_{ji} - \nabla_j K_{ki},$$

$$F_i^a \nabla_a K^*_{kj} = \nabla_j K_{ki} - \nabla_k K_{ji} \quad ,$$

$$\nabla_a K^a{}_h = \frac{1}{2} \nabla_h K, \qquad \nabla_a K^{*a}_k = \frac{1}{2} F_k^a \nabla_a K \ ,$$

we have

$$\nabla_t H_{kji}{}^t = \frac{n}{n+4} (\nabla_k K_{ji} - \nabla_j K_{ki}) + \frac{2b-nc}{2nc(n+4)} (g_{ji}\nabla_k K -$$

$$- g_{ki}\nabla_j K + F_{ji}F_k^a\nabla_a K - F_{ki}F_j^a\nabla_a K) + \frac{4n+4-6a+c}{(n+4)c} F_{kj}F_i^a\nabla_a K \ .$$

Now we assume that

$$\nabla_t H_{kji}{}^h = 0 \tag{3.3}$$

Then $\nabla_t H_{kji}{}^t = 0$ too, and the above equation reduces to

$$\nabla_k K_{ji} - \nabla_j K_{ki} =$$

$$= \frac{nc-2b}{2n^2c} (g_{ji}\nabla_k K - g_{ki}\nabla_j K + F_{ji}F_k^a\nabla_a K - F_{ki}F_j^a\nabla_a K)$$

$$- \frac{4n+4-6a+c}{nc} F_{kj}F_i^a\nabla_a K \ .$$

Transvecting the above equation with $F_r^i F_s^j$ and taking into account that

$$(\nabla_k K_{ji} - \nabla_j K_{ki})F_r^i F_s^j = \nabla_r K_{sk} \ ,$$

we have

$$\nabla_r K_{sk} = \frac{nc-2b}{2n^2c} (g_{rs}\nabla_k K - F_{rk}F_s^j\nabla_j K + F_{sr}F_k^a\nabla_a K + g_{kr}\nabla_s K) + \tag{3.4}$$

$$+ \frac{4n+4-6a+c}{nc} g_{ks}\nabla_r K \ .$$

At last, transvecting (3.4) with $g^{sk}$ and taking into account (2.22), we obtain

$$[n(n^2+4n-4) - a(n^2+2n-8) - 16]\nabla_r K = 0.$$

$$[n(n^2+4n-4) - a(n^2+2n-8) - 16] \neq 0$$

because of the assumption $a \neq n+2$. Therefore

$$\nabla_r K = 0. \tag{3.5}$$

THEOREM. *The necessary and sufficient condition that a Kähler space be the space with a parallel tensor* $H_{kji}{}^{h}$ *is that it be a locally symmetric space.*

P r o o f. From (3.4) and (3.5) we find

$$\nabla_i K_{jk} = 0.$$

Consequently, using (3.2) and (3.3), we get $\nabla_t K_{kji}{}^{h} = 0$. Con - versely, if the Kähler space is symmetric, the tensor $H_{kji}{}^{h}$ satisfies (3.3)

## 4. COMPLEX CONHARMONIC CONNECTION

In the paper |3|, Y. Ishii defined the conharmonic transformation as such a comformal transformation $\bar{g}_{ij} = e^{2\sigma} g_{ij}$ which changes a harmonic function A defined by

$$g^{ij}\nabla_i\nabla_j A = 0 \tag{4.1}$$

into a function

$$\bar{A} = e^{2k\sigma} A \tag{4.2}$$

satisfying

$$\bar{g}^{ij}\bar{\nabla}_i\bar{\nabla}_j\bar{A} = 0,$$

where k is a suitable constant and $\bar{\nabla}_i$ is the operator of the covariant differentiation with respect to

$$\overline{\left\{ {h \atop ji} \right\}} = \left\{ {h \atop ji} \right\} + \delta_j^h \sigma_i + \delta_i^h \sigma_j - g_{ij}\sigma^h .$$

The main purpose of the present section is to find, using the connection (1.2), the complex analogue of Ishii´s results.

We assume that condition (4.1) is satisfied and we seek the condition upon σ in order that the function (4.2) satisfies

$$\bar{g}^{ij} D_j D_i \bar{A} = 0. \tag{4.3}$$

First, we have

$$D_i\bar{A} \equiv \bar{A}_i = \partial_i\bar{A} = e^{2k\sigma}(2kA\sigma_i + A_i), \quad A_i = \partial_i A.$$

Then, taking into account (1.2), we get

$$D_j\bar{A}_i = e^{2k\sigma}[2kA\nabla_j\sigma_i + \nabla_j A_i + 4k(k-1)A\sigma_j\sigma_i + (2k-1)\sigma_j A_i +$$
$$+ (2k-1)\sigma_i A_j + 2kAg_{ij}\sigma^a\sigma_a + g_{ij}\sigma^a A_a$$
$$- 2kAF^a_j\sigma_a\kappa_i - 2kAF^a_i\sigma_a\kappa_j + 2kAF_{ji}\kappa^a\sigma_a$$
$$- F^a_j A_a\kappa_i - F^a_i A_a\kappa_j + F_{ji}\kappa^a A_a] .$$

Therefore

$$\bar{g}^{ij}D_j\bar{A}_i = e^{2(k-1)\sigma}[g^{ij}\nabla_j A_i + 2kAg^{ij}\nabla_j\sigma_i + 2k(2k-2+n)A\sigma_a\sigma^a$$
$$+ (4k-2+n)\sigma_a A^a - 4kAg^{ij}F^a_j\sigma_a\kappa_i - 2g^{ij}F^a_j\kappa_i A_a]$$

Consequently, using the conditions (4.1) and (4.3), we find

$$2kAg^{ij}\nabla_j\sigma_i + 2k(2k-2+n)A\sigma_a\sigma^a + (4k-2+n)\sigma_a A^a -$$
$$- 4kAg^{ij}F^a_j\sigma_a\kappa_i - 2g^{ij}F^a_j\kappa_i A_a = 0 ,$$

or

(4.4) $$2kAg^{ij}\nabla_j\sigma_i + 2k(2k+n)A\sigma_a\sigma^a + (2k+n)\sigma_a A^a = 0,$$

because of

$$-g^{ij}F^a_j\kappa_i A_a = \sigma^a A_a ,$$

and

$$-g^{ij}F^a_j\sigma_a\kappa_i = \sigma_a\sigma^a .$$

If we determine the number $k$ by $k = -\frac{n}{4}$, (4.4) reduces to

(4.5) $$2g^{rs}\nabla_r\sigma_s + n\sigma_a\sigma^a = 0 .$$

(4.5) is the required condition upon $\sigma$.

We call an affine connection (1.2) where the conditions (1.3) and (4.5) are satisfied a <u>complex conharmonic connection.</u>

The tensor formed with the curvature tensor of the complex conharmonic connection and independent of the function $\sigma$ can be called the complex conharmonic curvature tensor. On the other hand, the condition (1.7) reduces to the condition (4.5), if $a=\frac{n}{2}$ . It follows that (2.29) is the complex conharmonic curvature tensor. In other words, the tensor (2.28 ) generalizes the complex conharmonic curvature tensor, too.

## 5. GENERALIZATION OF THE CONTACT BOCHNER CURVATURE TENSOR

Now, we compute the curvature tensor $R_{kji}{}^{h}$ of the contact conformal connection (1.5) , and find |2|:

$$
\begin{aligned}
R_{kji}{}^{h} = K_{kji}{}^{h} & - (\delta_k^h - \eta_k\eta^h)p_{ji} + (\delta_j^h - \eta_j\eta^h)p_{ki} - p_k{}^h(g_{ji}-\eta_j\eta_i) + \\
& + p_j{}^h(g_{ki}-\eta_k\eta_i) - \phi_k{}^h q_{ji} + \phi_j{}^h q_{ki} - q_k{}^h\phi_{ji} + \\
& + q_j{}^h\phi_{ki} - \alpha_{ki}\phi_i^h - \phi_{kj}\beta_i{}^h + \phi_k{}^h\phi_{ji} - \\
& - \phi_j{}^h\phi_{ki} - 2\phi_{kj}\phi_i{}^h ,
\end{aligned} \tag{5.1.}
$$

where

$$p_{ji} = \nabla_j\sigma_i - \sigma_i\sigma_j + (\kappa_j-\eta_j)(\kappa_i-\eta_i) + \frac{1}{2}\sigma_a\sigma^a(g_{ji}-\eta_j\eta_i) ,$$

$$q_{ji} = \nabla_j\kappa_i - \sigma_j(\kappa_i-\eta_i) - \sigma_i(\kappa_j-\eta_j) + \frac{1}{2}\sigma_a\sigma^a\phi_{ji} ,$$

$$\alpha_{kj} = -(\nabla_k\kappa_j - \nabla_j\kappa_k), \quad \beta_i{}^h = (2\sigma_i\kappa^h - \kappa_i\sigma^h) .$$

Using the condition (1.7) where, now, $a \neq \frac{2(m^2-1)}{2m+1}$ , and proceding in a similar manner as in section 2, we find

$$
\begin{aligned}
p_{ji} = & -\frac{R_{ji}}{2(m+2)} + \frac{K_{ji}}{2(m+2)} + [E + (e^{2\sigma}\bar{R} - K)D]g_{ij} \\
& - [E - \frac{4}{2(m+2)} + (e^{2\sigma}\bar{R} - K)D]\eta_i\eta_j
\end{aligned} \tag{5.2}
$$

(5.3) $$q_{ji} = \frac{\bar{R}_{ja}\phi_i^a}{2(m+2)} - \frac{K_{ja}\phi_i^a}{2(m+2)} + [E + (e^{2\sigma}\bar{R} - K)D]\phi_{ji}$$

(5.4) $$\alpha_{ji} = -\frac{\bar{R}_{ja}\phi_i^a}{m+2} + \frac{K_{ja}\phi_i^a}{m+2} - [(2E + \frac{2m}{C}) + e^{2\sigma}(2D - \frac{1}{C})\bar{R} - (2D - \frac{1}{C})K]\phi_{ji} + A_{ji},$$

(5.5) $$\beta_{ji} = -\frac{\bar{R}_{ja}\phi_i^a}{m+2} + \frac{K_{ja}\phi_i^a}{m+2} + B_{ji} - [(2E + 2\frac{m+2-a}{C}) + e^{2\sigma}\bar{R}(2D - \frac{m+2-a}{mC}) - K(2D - \frac{m+2-a}{mC})]\phi_{ji},$$

where 2m+1 is the dimension of the considered Sasakian manifold, and

$$A_{ji} = \frac{(R_{rjis} - R_{isrj})\phi^{rs}}{2-2m},$$

$$B_{ji} = A_{ji} - \frac{M_{rsji}\phi^{rs}}{2m},$$

$$M_{rjih} = R_{kjih} - R_{ihkj} + \phi_{kh}\frac{(R_{rjis} - R_{isrj})\phi^{rs}}{2-2m} - \phi_{jh}\frac{(R_{rkis} - R_{isrk})\phi^{rs}}{2-2m} + \phi_{ji}\frac{(R_{rkhs} - R_{hsrk})\phi^{rs}}{2-2m} - \phi_{ki}\frac{(R_{rjhs} - R_{hsrj})\phi^{rs}}{2-2m}$$

$$\bar{R}_{ji} = R_{ji} - A_{ja}\phi_i^a - A_{ai}\phi_j^a - B_{ib}\phi_j^b - \frac{(R_{rs} - A_{as}\phi_r^a)\phi^{rs}}{} \phi_{ji}$$

$$R_{ji} = R_{aji}{}^a, \quad \bar{R} = \bar{R}_{ji}\bar{g}^{ji}$$

$$C = 4(1-m^2) + 2(2m+1)a$$

$$E = \frac{C - [2+2m-m^2+a(m-1)]}{(m+2)C}, \quad D = \frac{2+2m-m^2+a(m-1)}{2m(m+2)C}.$$

Substituting (5.2), (5.3), (5.4) and (5.5) into (5.1), we obtain that the tensor

$$\begin{aligned}
\mathcal{H}_{kji}{}^{h} &= \mathcal{R}_{kji}{}^{h} \\
&- (\delta_k^h - \bar{\eta}_k \bar{\eta}^h)\left[\frac{\bar{\mathcal{R}}_{ji}}{2(m+2)} - \bar{\mathcal{R}}(\bar{g}_{ij} - \bar{\eta}_i \bar{\eta}_j) D\right] + (\delta_j^h - \bar{\eta}_j \bar{\eta}^h)\left[\frac{\bar{\mathcal{R}}_{ki}}{2(m+2)} - \right. \\
&\left. - \bar{\mathcal{R}}(\bar{g}_{ki} - \bar{\eta}_k \bar{\eta}_i) D\right] - (\bar{g}_{ji} - \bar{\eta}_j \bar{\eta}_i)\left[\frac{\bar{\mathcal{R}}_k{}^h}{2(m+2)} - \bar{\mathcal{R}}(\delta_k^h - \bar{\eta}_k \bar{\eta}^h) D\right] + \\
&+ (\bar{g}_{ki} - \bar{\eta}_k \bar{\eta}_i)\left[\frac{\bar{\mathcal{R}}_j{}^h}{2(m+2)} - \bar{\mathcal{R}}(\delta_j^h - \eta_j \eta^h) D\right] + \phi_k{}^h\left(\frac{\bar{\mathcal{R}}_{ja}\phi_i^a}{2(m+2)} + \bar{\mathcal{R}}\bar{\phi}_{ji} D\right) - \\
&- \phi_j{}^h\left(\frac{\bar{\mathcal{R}}_{ka}\phi_i^a}{2(m+2)} + \bar{\mathcal{R}}\bar{\phi}_{ki} D\right) + \bar{\phi}_{ji}\left(\frac{\bar{\mathcal{R}}_{ka}\bar{\phi}^{ha}}{2(m+2)} + \bar{\mathcal{R}}\phi_j{}^h D\right) - \bar{\phi}_{ki}\left(\frac{\bar{\mathcal{R}}_{ja}\bar{\phi}^{ha}}{2(m+2)}\right. \\
&\left. + \bar{\mathcal{R}}\phi_j{}^h D\right) - \phi_j{}^h\left[2\frac{\bar{\mathcal{R}}_{ka}\phi_j^a}{2(m+2)} + \bar{\mathcal{R}}\left(2D - \frac{1}{C}\right)\bar{\phi}_{kj} - A_{kj}\right] \\
&- \bar{\phi}_{kj}\left[2\frac{\bar{\mathcal{R}}_{ia}\bar{\phi}^{ha}}{2(m+2)} + \bar{\mathcal{R}}\left(2D - \frac{m+2-a}{mC}\right)\phi_i{}^h - B_i{}^h\right]
\end{aligned}$$

is independent of the function $\sigma$ satisfying (1.7). In fact

$$\begin{aligned}
\mathcal{H}_{kji}{}^{h} &= K_{kji}{}^{h} - (\delta_k^h - \eta_k \eta^h)\left[\frac{K_{ji}}{2(m+2)} + (E-KD) g_{ji} - \left(E - \frac{2}{m+2} - KD\right)\eta_i \eta_j\right] \\
&+ (\delta_j{}^h - \eta_j \eta^h)\left[\frac{K_{ki}}{2(m+2)} + (E-KD) g_{ki} - \left(E - \frac{2}{m+2} - KD\right)\eta_k \eta_i\right] \\
(5.6)\quad &- (g_{ji} - \eta_j \eta_i)\left[\frac{K_k{}^h}{2(m+2)} + (E-KD)\delta_k^h - \left(E - \frac{2}{m+2} - KD\right)\eta_k \eta^h\right] \\
&+ (g_{ki} - \eta_k \eta_i)\left[\frac{K_j{}^h}{2(m+2)} + (E-KD)\delta_j^h - \left(E - \frac{2}{m+2} - KD\right)\eta_j \eta^h\right] \\
&- \phi_k^h\left[-\frac{K_{ja}\phi_i^a}{2(m+2)} + (E-KD)\phi_{ji}\right] + \phi_j{}^h\left[-\frac{K_{ka}\phi_i^a}{2(m+2)} + (E-KD)\phi_{ki}\right] \\
&- \phi_{ji}\left[-\frac{K_{ka}\phi^{ha}}{2(m+2)} + (E-KD)\phi_k{}^h\right] + \phi_{ki}\left[-\frac{K_{ja}\phi^{ha}}{2(m+2)} + (E-KD)\phi_j{}^h\right] \\
&- \frac{1}{m+2}(K_{ia}\phi^{ha}\phi_{kj} + K_{ka}\phi_j^a\phi_i{}^h) + \left[4E + \frac{4m+4-2a}{C} - \right. \\
&\left. - \left(4D - \frac{2m+2-a}{mC}\right)K\right]\phi_{kj}\phi_i{}^h + \phi_k{}^h\phi_{ji} - \phi_j{}^h\phi_{ki} - 2\phi_{kj}\phi_i{}^h .
\end{aligned}$$

If $a=2(m+1)$, then

$$C = 4(m+1)(m+2)\ , \qquad D = \frac{1}{8(m+1)(m+2)}\ ,$$

$$E = \frac{3m+4}{4(m+1)(m+2)}\ , \qquad E - \frac{2}{m+2} = -\frac{5m+4}{4(m+1)(m+2)}$$

and the tensor $H_{kji}{}^{h}$ has the form (1.6), i.e. $H_{kji}{}^{h}$ reduces to the contact Bochner curvature tensor. Therefore, the tensor $H_{kji}{}^{h}$ generalizes the contact Bochner curvature tensor. It generalizes the contact conharmonic curvature tensor, too.

In fact, we assume that condition (4.1) is satisfied and we seek the condition upon $\sigma$ in order that function (4.2) satisfies

$$\bar{g}^{ij}D_j D_i \bar{A} = 0,$$

where $D_i$ is the operator of the covariant differentiation with respect to the contact conformal connection (1.5). Proceeding in a similar manner as in section 4, we find that the required condition is:

$$\text{(5.7)} \qquad \nabla_a \sigma^a + \frac{m^2+m+1}{m}\sigma_a\sigma^a = 0$$

We shall an affine connection (1.5), where the condition (5.7) is satisfied, a contact conharmonic connection. The tensor formed with the curvature tensor of the contact conharmonic connection and independent of the function $\sigma$ can be called the contact conharmonic curvature tensor. On the other hand, the condition (5.7) is the special case of the condition (1.7). Therefore, the tensor (5.6) where $a = \frac{m^2+m+1}{m}$ is the contact conharmonic curvature tensor.

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REZIME

## JEDNO UOPŠTENJE BOHNEROVOG I KONTAKTNO-BOHNEROVOG TENZORA KRIVINE

U § 2 posmatra se Kähler-ov prostor koji dopušta takvu skalarnu funkciju $\sigma$ da je zadovoljen uslov (1.7). Koristeći tenzor krivine kompleksne konformne koneksije (1.2), konstruisan je tenzor $H_{kji}{}^{h}$ tj. tenzor (2.28), koji ne zavisi od funkcije $\sigma$. U slučaju kad je $a = n+2$, tenzor $H_{kji}{}^{h}$ se svodi na Bohner-ov tenzor. U § 3 dokazane su neke teoreme o tenzoru $H_{kji}{}^{h}$. U § 4 odredjen je tenzor kompleksno konharmonijske krivine koji je takodje jedan specijalan slučaj tenzora $H_{kji}{}^{h}$.

U § 5 posmatra se (2m+1)-dimenzioni prostor Sasaki-ja koji dopušta takvu skalarnu funkciju $\sigma$ da je zadovoljen uslov (1.7) uz ograničenje $a \neq \frac{2(m^2-1)}{2m+1}$. Koristeći tenzor krivine kontaktne konformne transformacije (1.5), konstruisan je tenzor $\mathcal{H}_{kji}{}^{h}$ tj. tenzor (5.6). Taj tenzor je uopštenje i kontaktno-Bochner-ovog tenzora krivine i tenzora kontaktno-konharmonijske krivine.