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ON A STRUCTURE DEFINED BY A TENSOR FIELD f OF THE TYPE (1,1) SATISFYING $f^{2\cdot 2^q+1}-f=0$

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ABSTRACT

In this paper was chose an adapted frame for f(2k+1,-1)-structure and a matrix of tensors g_{ij} and f_i^j with respect to this adapted frame. Given is the necessary and sufficient condition for an n-dimensional manifold M^n to admit a tensor field f of the type (1,1) and the rank r such that $f^{2\cdot 2^q+1} - f = 0$, $f^{2i+1}-f\neq 0$ for $1\leq i < 2^q$, $q\in N$.

1. Let us first observe the structure f with satisfies the condition $f^{2k+1} - f = 0$.

DEFINITION 1. *Let M^n be a differentiable manifold of the class C^∞ and let there be given a tensor field $f\neq 0$ of the type (1,1) and of the class C^∞ such that*

(1.1) $$f^{2k+1} - f = 0, \quad f^{2i+1} - f \neq 0 \quad \text{for} \quad 1\leq i<k$$

where k is a fixed positive integer greater than 1. Let rank f = r be constant. We call such a structure an (2k+1,-1)-structure or an f-structure of rank r and of 2k+1 -degree.

THEOREM 1.1. *For a tensor field f, $f\neq 0$ satisfying (1.1), the operators*

(1.2) $$m = 1 - f^{2k} \quad \text{and} \quad \ell = f^{2k}$$

(1.1) *denoting the identity operator applied to the tangent space at a point of the manifold, are complementary projection operators.*

P r o o f. We have

$$\ell + m = 1, \quad \ell^2 = \ell, \quad m^2 = m, \quad m\ell = \ell m = 0$$

by virtue of (1.1) which proves the Theorem.

Let L and M be the complementary distributions corresponding to the operators ℓ and m, respectively. If the rank $f = r$ is constant, then $\dim L = r$ and $\dim M = n - r$.

THEOREM 1.2. *For f satisfying* (1.1) *and* ℓ, m, *defined by* (1.2) *we have*

$$\ell f = f\ell = f, \quad mf = fm = 0, \quad f^2 m = 0.$$

P r o o f. Trivial.

THEOREM 1.3. *For f satisfying* (1.1) *and* m, *defined by* (1.2) *we have*

(1.3) $$(m+f^k)^2 = 1, \quad fm = mf = 0 .$$

P r o o f. Trivial.

THEOREM 1.4. *Suppose that there is given on* M^n *a projection operator* m *and that there exists a tensor field* f *such that* (1.3) *is satisfied, then* f *satisfies* (1.1).

PROPOSITION 1.1. *Let an* f*-structure of rank* r *and of the* 2k+1 *-degree be given on* M^n, *then* $f^{2k}\ell = \ell$, *and* $f^{2k}m = 0$. *Then* f^k *acts on* L *as almost a product structure operator and on* M *as almost a tangent structure operator.*

2. We shall now introduce a local coordinate system in the manifold and denote by f_i^p, ℓ_i^p, m_i^p the local components of the tensor f, ℓ, m, respectively. We shall also introduce a positive definite Riemannian metric g in the manifold and take r as mutually orthogonal unit vector u_a^p $(a,b,c = 1,2,\dots,r)$ in L and $n-r$ as mutually orthogonal unit vector u_A^p $(A,B,C = r+1,\dots,n)$

in M. We have

$$\ell^p_i u^i_b = u^p_b,\ \ \ell^p_i u^i_B = 0,\ \ m^p_i u^i_b = 0,\ \ m^p_i u^i_B = u^p_B .$$

We also have $f^p_i u^i_B = 0$. If we denote by (v^a_i, v^A_i) the matrix inverse to $(u^p_b,\ u^p_B)$, then v^a_i and v^A_i are both components of linearly independent covariant vectors and satisfy

$$(2.1)\qquad v^a_i u^i_b = \delta^a_b,\ \ v^a_i u^i_B = 0,\ \ v^A_i u^i_B = 0$$

$$v^A_i u^i_B = \delta^A_B,\ \ v^a_u u^p_a + v^A_i u^p_A = \delta^p_i .$$

We can easily prove that

$$(2.2)\qquad \ell^p_i v^a_p = v^a_i,\ \ \ell^p_i v^a_p = 0,\ \ m^p_i v^\ell_p = 0$$

$$m^p_i v^A_p = v^A_i \ .$$

From $mf = 0$, we find $f^p_i v^A_p = 0$. From $\ell^p_i u^i_a = u^p_a$, we find $\ell^p_i = v^a_i u^p_a$. From (2.1) and (2.2), we also find $m^p_i = v^A_i u^p_a$.

If we put $a_{ij} = v^a_j v^a_i + v^A_j v^A_i$, then a_{ij} is a globally positive definite Riemannian metric with respect to which $(u^p_b$, $u^p_B)$ forms an orthogonal frame such that $v^a_j = a_{ij} u^i_A$ and $v^A_j = a_{ij} u^i_A$.

If we put $\ell_{ij} = \ell^t_j a_{ti}$, $m_{ij} = m^t_j a_{ti}$, we find $\ell_{ji} = v^a_j v^a_i$, $m_{ji} = v^A_j v^A_i$. Consequently $\ell_{ji} + m_{ji} = a_{ji}$.

We can easily verify the following relations:

$$\ell^t_j \ell^s_i a_{ts} = \ell_{ji},\ \ \ell^t_j m^s_i a_{ts} = 0,\ \ m^t_j m^s_i a_{tj} = m_{ji} .$$

For any two vectors x,y with components x^i, y^i, let us put

$$m^*(x,y) = m_{st} x^s y^t,\ \ a(x,y) = a_{st} x^s y^t ,$$

$$g(x,y) = \frac{1}{K}\ (a(x,y) + (\sum_{z=1}^{2k-1} (f^z x, f^z y))a + m^*(x,y)) .$$

Then we have

$$m^*(u_A,u_a) = a(u_A,u_a) = a(fu_A,fu_a) = a(f^2u_A,f^2u_a) =$$

$$= \ldots = a(f^{2k-1}u_A, f^{2k-1}u_a) = 0,\ \ g(u_A,u_a) = 0.$$

Thus L and M are orthogonal with respect to g. We also have

$$m^*(u_a,u_b)=0, \quad a(f^{2k}u_a,f^{2k}u_b)=a(u_a,u_b) \ .$$

Hence

$$g(u_a,u_b)=\frac{1}{k}\ (a(u_a,u_b)+(\sum_{z=1}^{2k-1}(f^z u_a,f^z u_b))a) \ ,$$

$$g(fu_a,fu_b)=\frac{1}{k}\ (a(\sum_{z=1}^{2k}(f^z u_a,f^z u_b)))=$$

$$=\frac{1}{k}\ (a(\sum_{z=1}^{2k-1}(f^z u_a,f^z u_b))+a(u_a,u_b).$$

That is

$$g(x,y) = g(fx,\ fy)$$

for all vectors x,y in L.

We assume that $f^i_L = f^i \ell/L$ $(i<2k)$ is not the identity operator of L. Then f_L is a linear transformation of L with minimal polynominal $x^{2k}-1=0$. (We know that $f^{2k}=1$ on L). The polynominal $(x^k-1)(x^k+1)=0$ has simple roots

$$e^{\frac{2\pi i}{k}},e^{3\frac{2\pi i}{k}},\dots,e^{(2k-1)\frac{2\pi i}{k}},e^{2\frac{2\pi i}{k}},e^{4\frac{2\pi i}{k}},\dots,e^{2k\frac{2\pi i}{k}}$$

The eigen vectors which correspond to these eigenvalues are e_1, $e_3,\dots,e_{2k-1}$, e_2, $e_4,\dots,e_{2k}$, respectively. Let us denote by L_2 the vector space generazed by vectors $e_1,e_3,\dots,e_{2k-1}$ and by L_1 the vector space generized by vectors $e_2,e_4,\dots,e_{2k}$.

$$f^k=-1 \ \text{ on } \ L_2, \quad f^k=1 \ \text{ on } \ L_1 \ .$$

For $x\in L_1$ and $y\in L_2$, we have

$$g(x,y)=g(fx,fy)=g(f^k x,f^k y)=g(x,-y)=-g(x,y) \ .$$

Hence L_1, L_2 are orthogonal with respect to the metric g.

We assume that $f^j\neq 1$ on L_1, $j<k$ and $f^j\neq -1$ on L_2, $j<k$.

Then f is a linear transformation of L_2 with the minimal polynomial $x^k+1=0$, with the eigenvalue $\sqrt[k]{-1}$, to which correspond the eigen vector $e_1', e_2',\dots,e_k'$ and $L_2=L_2^1+L_2^2+\dots+L_2^k$ where L_2^s are subspaces of L_2 generized by the vectors e_s' .

It is also an f linear transformation on L_1 with the minimal polynomial $x^k - 1 = 0$, with the eigenvalue $\sqrt[k]{1}$, to which correspond the eigen vectors $e'_{k+1}, e'_{k+2}, \dots, e'_{2k}$. Now, $L_1 = L_1^{k+1} + L_1^{k+2} + \dots + L_1^{2k}$, where L_1^{k+1} are subspaces of L_1 generized by the vectors e'_{k+s} .

L_1^{k+p} and L_1^{k+r} are orthogonal with respect to g only if $k = 2^q$, $q \in N$, which is then shown by induction. In the following text $k = 2^q$, $q \in N$.

Let $x \in L_2$. Then $f(x) \in L_2$. We have

$$g(x,fx) = g(fx,f^2x) = \dots = g(f^kx,f^{k+1}x) = g(-x,fx) = -g(x,fx),$$

Hence $L_2 = \bar{L}_2 \oplus f(\bar{L}_2)$ and $\bar{L}_2$ and fL_2 are orthogonal spaces with respect to g.

We assume that $\dim L_2 = d$, $\dim f\bar{L}_2 = s$, then

$$\dim L_2 = d+s = 2p, \quad \dim L_1 = r - d - s = r - 2p. \text{ Let } r = (q+2)2p.$$

In |4| the following Theorem is proved:

THEOREM. *If*

$$f^{\bar{k}} = \begin{bmatrix} 0 & E_p \\ -E_p & 0 \end{bmatrix}$$

then $\bar{k} \leq p$ *and* p *is divisible by* $\bar{k}$, $(p = s\bar{k})$.

In our case, there is such a state in the space L_2. $(d+s = 2p)$.

Let $u_1, \dots, u_{2p}$ be an orthogonal basis of L_2 $(p = s \cdot 2^{q-1})$, and $u_{2p+1}, u_{2p+2}, \dots, u_{r-2p}$ be an orthogonal basis of L_1, both with respect to g, then $u_1, \dots, u_{2p}, u_{2p+1}, \dots, u_{r-2p}$ is an orthogonal basis of L such that

$$\text{for:} \quad L_2 : f(u_i) = u_{i+\frac{p}{2^q}}, \; f(u_{i+2p-\frac{p}{2^q}}) = -u_i, \quad i = 1,2,\dots,p$$

for L_1:
$$\begin{cases} f(u_{2p+1}) = u_{2p+i+\frac{p}{2^{q-1}}},\ f(u_{4p+i-\frac{p}{2^{q-1}}}) = -u_{2p+i}, & i=1,2,\dots,p \\ & i=1,2,\dots,p \\ f(u_{4p+i}) = u_{4p+i+\frac{p}{2^{q-2}}},\ f(u_{6p+i-\frac{p}{2^{q-2}}}) = -u_{4p+i}, & i=1,2,\dots,p \\ & i=1,2,\dots,p \\ \vdots & \\ f(u_{2qp+i}) = u_{2pq+i},\ f(u_{2(q+1)p-i}) = -u_{2p(q+1)-i} & \\ & i=1,2,\dots,p\ . \end{cases}$$

Next, we choose in M an orthogobal basis $u_{r+1},\dots,u_n$ with respect to g. Then with respect to the orthogonal frame $u_1,\dots,u_n$ the tensors g_{ij} and f_i^j have the components

(2.3)
$$f = \begin{bmatrix} 0 & E_{2p-\frac{2p}{2^q}} & & & & & & & & \\ -E_{\frac{2p}{2^q}} & 0 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 0 & E_{2p-\frac{p}{2}} & & & & & \\ & & & -E_{\frac{p}{2}} & 0 & & & & & \\ & & & & & 0 & E_p & & & \\ & & & & & -E_p & 0 & & & \\ & & & & & & & E_p & 0 & \\ & & & & & & & 0 & -E_p & \\ & & & & & & & & & 0_{n-r} \end{bmatrix} \Bigg\} q+1 \qquad g = \begin{bmatrix} E_{2p} & & & \\ & \ddots & & \\ & & E_{2p} & \\ & & & E_{n-r} \end{bmatrix} \Bigg\} q+1$$

We call such a frame an adapted frame of f(2k+1,-1) structure.

Let $\bar{u}_1,\dots,\bar{u}_n$ be another adapted frame with respect to which the metric tensor g and the tensor f have the same components as (2.3). We put $\bar{u}_i = \gamma_i^j u_j$ then we can find that γ has the form

(2.4)

$$
\gamma = \begin{bmatrix} S_{(\frac{2p}{2^q})} & & & & & & & \\ & S_{(\frac{2p}{2^{q-1}})} & & & & & & \\ & & \ddots & & & & & \\ & & & S_{(\frac{p}{2})} & & & & \\ & & & & A_p & B_p & & \\ & & & & -B_p & A_p & & \\ & & & & & & 0_{2p} & \\ & & & & & & & 0_{n-r} \end{bmatrix}
$$

where $S_{(\frac{2p}{i})}$, $i=2,4,\dots,2^q$ is a matrix of format $2p$ and has the form

(2.5)

$$
S_{(\frac{2p}{i})} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & \dots & A_i \\ -A_i & A_1 & A_2 & A_3 & \dots & A_{i-1} \\ -A_{i-1} & -A_i & A_1 & A_2 & \dots & A_{i-2} \\ \vdots & & & & & \\ -A_3 & -A_4 & -A_5 & & & A_2 \\ -A_2 & -A_3 & -A_4 & & \dots & A_1 \end{bmatrix}
$$

where each matrix A_t, $t=1,\dots,i$ has a format $(\frac{2p}{i}) \times (\frac{2p}{i})$, i.e. $s \times s$.

Let $\bar{S}_{(\frac{2p}{i})}$ be the tangent group defined by $S_{(\frac{2p}{i})}$. Then we can say that the group of the tangent bundle of the manifold can be reduced to

$$
\bar{S}_{(\frac{2p}{2^q})} \times \bar{S}_{(\frac{2p}{2^{q-1}})} \times \dots \times \bar{S}_{(\frac{2p}{4})} \times U_p \times O_{2p} \times O_{n-r}
$$

THEOREM 2.1. *A necessary and sufficient condition for an* n*-dimensional manifold* M^n *to admit a tensor field* $f \neq 0$ *of the type* (1,1) *and of rank* r *, such* $f^{2\cdot 2^q+1} - f = 0$*, is that*

i) $r=(q+1)2p$, ii) $p=s\cdot 2^q=s\cdot k$ *and* iii) *the group of the tangent bundle of the manifild be reduced to the group*

$$\bar{S}_{(\frac{2p}{2^q})}\times\bar{S}_{\frac{2p}{2^{q-1}}}\times\ldots\times\bar{S}_{(\frac{2p}{4})}\times U_p\times O_{2p}\times O_{n-r}\,.$$

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REZIME

O STRUKTURI KOJA JE DEFINISANA TENZORSKIM POLJEM f TIPA (1,1) KOJE ISPUNJAVA USLOV $f^{2\cdot 2^q+1}-f=0$

U radu je definisana $f(2k+1,-1)$ struktura, izabran adaptirani reper za tu strukturu, nadjene su martice za tenzore g_{ij} i f_i^j u odnosu na taj adaptirani reper. Dat je potreban i dovoljan uslov da se n-dimenzionalna mnogostrukost može snabdeti tenzorskim poljem f tipa (1,1) i ranga r da je $f^{2\cdot 2^q+1}-f=0$, $f^{2i+1}-f\neq 0$ za $1\leq i<2^q$, $q\in N$.

TEOREMA 2.1. *Potreban i dovoljan uslov da se n-dimenzionalna mnogostrukost M^n može snabdeti tenzorskim poljem $f\neq 0$ tipa (1,1) i ranga r, tako da je $f^{2\cdot 2^q+1}-f=0$ je*

i) $r = (q+1)2p$

ii) $p = s\cdot 2^q = sk$

iii) *grupa tangentnog bandla mnogostrukosti se reducira do*

$$\bar{S}_{(\frac{2p}{2^q})} \times \bar{S}_{(\frac{2p}{2^{q-1}})} \times \dots \times \bar{S}_{(\frac{2p}{4})} \times U_{(p)} \times O_{(2p)} \times O_{(n-r)}$$

gde je $\bar{S}_{(\frac{2p}{i})}$ *tangentna grupa definisana sa* $S_{(\frac{2p}{i})}$ *čiji je oblik dat formulom* (2.5).