Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 12 (1982)
Review of Research Faculty of Science-University of Novi Sad, Volume 12(1982)

WEYL-ŌTSUKI SPACES OF THE SECOND KIND WITH A SPECIAL TENSOR P

Nevena Pušić
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

ABSTRACT

In this paper, we define Weyl-Ōtsuki spaces of the second kind and order n (denoted $SW\text{-}O_n$) with a tensor P in the form $P_s^i = S(x)\delta_s^i$, where S(x) is a non-zero and non-linear C^r $(r \geq 3)$ function. We establish some relations between the curvature tensor and the conformal curvature-like tensor in $SW\text{-}O_n$ and analogous objects in an adjoint Riemannian space.

1. The basic objects of spaces investigated by T.Ōtsuki (|1|) are two objects of connection, denoted by $'\Gamma_{jk}^i$ and $''\Gamma_{jk}^i$, and a regular tensor field P of the type (1,1). $'\Gamma_{jk}^i$ and $''\Gamma_{jk}^i$ are objects of two affine connections $'\Gamma$ and $''\Gamma$ named the contravariant and covariant part of the regular general connection respectively.

The basic covariant differentiation is defined by the following formulas

$$v_{i|k} = \frac{\partial v_i}{\partial u^k} - {}''\Gamma_{ik}^s v_s$$

$$v^i_{|k} = \frac{\partial v^i}{\partial u^k} + {}'\Gamma_{sk}^i v^s$$

The covariant differentiation with respect to the general regular connection is defined for an arbitrary tensor in the following way:

$$(1.1)\qquad T^i_{jk,\ell} = P^i_a T^a_{bc|\ell} P^b_j P^c_k \ .$$

Connections $'\Gamma$ and $''\Gamma$ are mutually dependent by relation

$$(1.2)\qquad Q^i_{j|k} = 0$$

where Q denotes the inverse of P.

The Weyl-Ōtsuki space (W-O_n) is defined by A.Moór (|2|, |3|). This is an Ōtsuki space provided with a symmetric positive metric tensor g_{ij} and a recurrence vector γ_k. The W-O_n space satisfied the following conditions:

a) $g_{ij,k} = \gamma_k g_{ij}$ (the metric tensor is recurrent),

b) $''\Gamma^i_{jk} = ''\Gamma^i_{kj}$ (the covariant part of a general regular connection is symmetric),

c) $P_{ij} = g_{ia}P^a_j = g_{ja}P^a_i = P_{ji}$.

In W-O_n spaces, the coefficients of connection $''\Gamma$ have the form

$$''\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} - \frac{1}{2} g^{is}(\gamma_j g_{ab} Q^a_s Q^b_k + \gamma_k g_{ab} Q^a_s \; Q^b_j - \gamma_s g_{ab} Q^a_j Q^b_k)$$

where $\left\{ {i \atop jk} \right\}$ denote the Christoffel symbols with respect to g_{ij}.

M.Prvanović (|4|,|5|) has defined another kind of metric Ōtsuki space with a recurrent metric tensor; these Ōtsuki spaces with a regular general connection satisfy the following conditions :

a') $g_{ij,k} = \gamma_k m_{ij}$, where m_{ij} is a symmetric tensor field,

b') the connection $'\Gamma$ is symmetric ($'\Gamma^i_{jk} = {'\Gamma}^i_{kj}$),

c') $P_{ij} = g_{ia}P^a_j = g_{ja}P^a_i = P_{ji}$.

The coefficients of connections $''\Gamma$ and $'\Gamma$ can be calculated by the following formulas:

$$(1.3)\qquad \begin{aligned} ''\Gamma^i_{jk} &= {''\overset{m}{\Gamma}}{}^i_{jk} + \frac{1}{2} g^{is}(\gamma_q m_{pk} Q^q_s Q^p_j - \gamma_k m_{pq} Q^p_s Q^q_j - \gamma_t m_{pk} Q^p_s Q^t_j) \\ '\Gamma^i_{jk} &= {'\overset{m}{\Gamma}}{}^i_{jk} + \frac{1}{2} g^{st}(\gamma_q m_{jk} Q^q_s Q^i_t - \gamma_k m_{pj} Q^p_s Q^i_t - \gamma_j m_{pk} Q^p_s Q^i_t) \end{aligned}$$

where ${}''\overset{m}{\Gamma}{}^i_{jk}$ and ${}'\overset{m}{\Gamma}{}^i_{jk}$ denote the coefficients of the covariant and contravariant part of the metric general regular connection ($g_{ij,k}=0$, $\gamma_k=0$). These coefficients have the form

$$(1.4)\quad \begin{aligned} {}''\overset{m}{\Gamma}{}^i_{jk} &= \{{}^{\;i}_{jk}\} + \overset{\circ}{\nabla}_{[a}P^i_{k]}Q^a_j - \overset{\circ}{\nabla}_{[a}P^{\ell}_{k]}Q^{ai}g_{j\ell} - \overset{\circ}{\nabla}_{[a}P^{\ell}_{q]}Q^{ai}P_{k\ell}Q^q_j \\ {}'\overset{m}{\Gamma}{}^i_{jk} &= \{{}^{\;i}_{jk}\} + \nabla_{(k}P^a_{j)}Q^i_a - \overset{\circ}{\nabla}_{[a}P^{\ell}_{k]}Q^{at}Q^i_tP_{j\ell} - \overset{\circ}{\nabla}_{[a}P^{\ell}_{j]}Q^{at}Q^i_tP_{k\ell} \end{aligned}$$

The Weyl-Ōtsuki space of the second kind is defined as a metric Ōtsuki space, which satisfies the conditions a´), b´) and c´) for $m_{ij}=P_{ij}$.

2. In this paper we define the Weyl-Ōtsuki spaces of the second kind as metric Ōtsuki spaces which satisfy the conditions a´), b´) and c) and $m_{ij}=\overset{n}{P}_{ij}=P_{ia_1}P^{a_1}_{a_2}\dots P^{a_{n-1}}_j$. Such an Ōtsuki space is a Weyl-Ōtsuki space of the second kind and the order n. In particular, if $m_{ij}=P_{ij}$, it is a Weyl-Ōtsuki space of the second kind and order 1. Now, we put

$$(2.1)\qquad P^i_j = S(x)\delta^i_j$$

where $S(x)$ is a real C^r ($r\geq 3$) non-zero function: at least one of the second partial derivatives of function $S(x)$ is different from 0.

Taking into account (1.2), we can obtain

$$(2.2)\qquad {}'\Gamma^i_{jk} = {}''\Gamma^i_{jk} + \frac{S_k}{S}\,\delta^i_j \quad \left(S_k = \frac{\partial S(x)}{\partial u^k}\right).$$

We denote a Weyl-Ōtsuki space of the second kind and order n with tensor P in the form (2.1) with $SW\text{-}O_n$.

It is easy to see that connections $'\Gamma$ and $''\Gamma$ in $SW\text{-}O_n$ have the same curvature tensor, named "the curvature tensor of $SW\text{-}O_n$".

3. If the general regular connection in $SW\text{-}O_n$ is metric, we can get, using (1.4),

$$(3.1)\quad \begin{aligned} \overset{m}{''\Gamma}{}^{i}_{jk} &= \left\{{i \atop jk}\right\} + \frac{1}{S}\,(S_j\delta^i_k - S^i g_{jk}) \\ \overset{m}{'\Gamma}{}^{i}_{jk} &= \left\{{i \atop jk}\right\} + \frac{1}{S}\,(S_j\delta^i_k + S_k\delta^i_j - S g_{jk}). \end{aligned}$$

To calculate the curvature tensor of $SW\text{-}O_n$ with the metric general regular connection, we put

$$\overset{m}{''\Gamma}{}^{i}_{jk} = \left\{{i \atop jk}\right\} + H^i_{jk}$$

where H^i_{jk} is $\lambda_j\delta^i_k - \lambda^i g_{jk}$ and λ_j stands for $\frac{S_i}{S}$. We obtain, from the well-known formula for the coefficients of the covariant curvature tensor in $SW\text{-}O_n$:

$$(3.2)\quad \begin{aligned} \overset{m}{''R}_{irkj} &= \overset{m}{'R}_{irkj} = \overset{m}{R}_{irkj} = \\ &= K_{irkj} + g_{rk}\psi_{ji} - g_{rj}\psi_{ki} + g_{ij}\psi_{kr} - g_{ik}\psi_{jr} \end{aligned}$$

where ψ_{ki} denotes $\overset{o}{\nabla}_k\lambda_i - \lambda_i\lambda_k + 1/2\lambda^s\lambda_s g_{ik}$ and λ_i stands for $\frac{S_i}{S}$. K_{irkj} denotes the corresponding component of the Riemann-Christoffel curvature tensor, with respect to Christoffel symbols.

We can see that all ψ_{ki} are symmetric. Hence, we can easily get the next relations for components of the curvature tensor in $SW\text{-}O_n$:

$$(3.3)\quad \overset{m}{R}_{irkj} = -\overset{m}{R}_{irkj};\ -\overset{m}{R}_{rijk};\quad \overset{m}{R}_{irkj} = \overset{m}{R}_{kjir}$$

Transvecting (3.2) by g^{ij}, we obtain

$$(3.4)\quad \overset{m}{R}_{rk} = K_{rk} + (n-2)\psi_{kr} + g_{kr}\psi_{ji}g^{ji}$$

where $\overset{m}{R}_{rk}$ denotes $\overset{m}{R}_{irkj}g^{ij}$ and K_{rk} is the corresponding component of the Ricci tensor with respect to Christoffel symbols.

Now, we have to calculate $\psi_{ji}g^{ji}$. We transvect (3.4) by g^{rk} and get

$$\overset{m}{R} = K + 2(n-1)\psi_{ji}g^{ji}$$

$$(3.5)\quad \psi_{ji}g^{ji} = \frac{\overset{m}{R} - K}{2(n-1)}$$

where $\overset{m}{R}$ denotes $\overset{m}{R}_{rk}g^{rk}$ and K is $K_{rk}g^{rk}$. Substituting (3.5) in (3.4), we get

$$\overset{m}{R}_{rk} = K_{rk} + (n-2)\psi_{kr} + g_{kr}\frac{\overset{m}{R} - K}{2(n-1)}$$

$$\psi_{kr} = \frac{K_{rk}-\overset{m}{R}_{rk}}{2-n} + \frac{K - \overset{m}{R}}{2(1-n)(2-n)} \tag{3.6}$$

Taking into account (3.6), we can express (3.2) as follows

$$\overset{m}{R}_{irkj} + \frac{1}{2-n}(g_{ij}\overset{m}{R}_{rk} - g_{ik}\overset{m}{R}_{rj} - g_{rj}\overset{m}{R}_{ik} + g_{rk}\overset{m}{R}_{ij}) + \tag{3.7}$$
$$+ \frac{\overset{m}{R}}{(n-1)(n-2)}(g_{ij}g_{kr} - g_{ik}g_{rj}) =$$
$$= K_{irkj} + \frac{1}{2-n}(g_{ij}K_{rk} - g_{ik}K_{rj} - g_{rj}K_{ik} + g_{rk}K_{ij}) +$$
$$+ \frac{K}{(n-1)(n-2)}(g_{ij}g_{kr} - g_{ik}g_{rj})$$

For every $SW\text{-}O_n$ space, there exists a uniquely determined Riemannian space with the same metric tensor. We call such a Riemannian space an adjoint Riemannian space for the space $SW\text{-}O_n$.

So, we have proved the following.

THEOREM 1. *In the $SW\text{-}O_n$ space with the metric regular general connection, the tensor on the left-hand side of (3.7) is equal to the conformal curvature tensor of the adjoint Riemannian space.*

COROLLARY 1. *If the adjoint Riemannian space of SW O_n is conformally Euclidean, the tensor on the left-hand side of (3.7) is equal to 0.*

4. If γ_k is different from zero and if $SW\text{-}O_n$ is of the order t, we obtain, using (1.3)

$$''\Gamma^i_{jk} = ''\overset{m}{\Gamma}{}^i_{jk} + \frac{S^{t-2}}{2}(\gamma^i g_{jk} - \gamma_k\delta^i_j - \gamma_j\delta^i_k)$$
$$'\Gamma^i_{jk} = '\overset{m}{\Gamma}{}^i_{jk} + \frac{S^{t-2}}{2}(\gamma^i g_{jk} - \gamma_k\delta^i_j - \gamma_j\delta^i_k) \tag{4.1}$$

The connection $"\Gamma$ is semi-symmetric, but $'\Gamma$ is symmetric and these two connections have the same curvature tensor. Thus, we have

$$'\Gamma^i_{jk} = \overset{m}{'\Gamma}{}^i_{jk} + H^i_{jk}$$

$$H^i_{jk} = \lambda^i g_{jk} - \lambda_k \delta^i_j - \lambda_j \delta^i_k$$

$$\lambda_k = \frac{S^{t-2}}{2} \gamma_k .$$

It is easy to prove this relation for the curvature tensor

$$R^i_{rkj} = \overset{m}{R}{}^i_{rkj} + \overset{m}{\nabla}_k H^i_{rj} - \overset{m}{\nabla}_j H^i_{rk} + H^s_{rj} H^i_{sk} - H^s_{rk} H^i_{sj}$$

where $\overset{m}{\nabla}$ denotes the classical covariant differentiation with respect to the coefficients of connection $\overset{m}{'\Gamma}$. The metric tensor is recurrent respect to $\overset{m}{\nabla}$ $(\overset{m}{\nabla}_k g_{ij} = \Pi_k g_{ij}$, where Π_k stands for $\frac{-2S_k}{S}$).

For the covariant components of the curvature tensor, we get

$$\begin{aligned} R_{irkj} = \overset{m}{R}_{irkj} &+ g_{rj}(\lambda_i \lambda_k - \lambda_i \Pi_k + \overset{m}{\nabla}_k \lambda_i) + \\ &+ g_{ik}(\lambda_r \lambda_j + \overset{m}{\nabla}_j \lambda_r) - g_{rk}(\lambda_i \lambda_j - \lambda_i \Pi_j + \overset{m}{\nabla}_j \lambda_i) - \\ &- g_{ij}(\lambda_r \lambda_k + \overset{m}{\nabla}_k \lambda_r) - g_{ir}(\overset{m}{\nabla}_k \lambda_j - \overset{m}{\nabla}_j \lambda_k) + \\ &+ \lambda^s \lambda_s (g_{rk} g_{ij} - g_{rk} g_{ik}). \end{aligned} \tag{4.2}$$

For the sum $R_{irkj} + R_{rijk}$, we can easily get

$$R_{irkj} + R_{rijk} = \overset{m}{R}_{irkj} + \overset{m}{R}_{rijk} + g_{rj}\theta_{ik} - g_{ij}\theta_{rk} + g_{ik}\theta_{rj} - g_{rk}\theta_{ij} \tag{4.3}$$

where

$$\theta_{ij} = 2\lambda_i \lambda_j - \lambda_i \Pi_j + 2\overset{m}{\nabla}_j \lambda_i - \lambda^s \lambda_s g_{ij} .$$

If $\gamma_k \neq 0$, the components of the curvature tensor do not satisfy the relations analogous to (3.3).

Then, we transvect (4.3) by g^{ij} and we obtain

$$(4.4)\qquad A_{rk} = \overset{m}{A}_{rk} + (2-n)\theta_{rk} - g_{rk}\theta_{ij}g^{ij}$$

where A_{rk}, $\overset{m}{A}_{rk}$ stand for $(R_{irkj} + R_{rijk})g^{ij}$ and $(\overset{m}{R}_{irkj} + \overset{m}{R}_{rijk})g^{ij}$, respectively. Next, we transvect (4.4) by g^{rk} and we get

$$(4.5)\qquad B = \overset{m}{B} + 2(n-1)\theta_{rk}g^{rk}$$

where B, $\overset{m}{B}$ stand for $A_{rk}g^{rk}$, $\overset{m}{A}_{rk}g^{rk}$, respectively. Hence, from (4.5) we obtain

$$(4.6)\qquad \theta_{rk}g^{rk} = \frac{B - \overset{m}{B}}{2(1-n)}$$

and, substituting (4.6) in (4.4),

$$(4.7)\qquad \theta_{rk} = \frac{A_{rk} - \overset{m}{A}_{rk}}{2-n} + \frac{B - \overset{m}{B}}{2(n-1)(n-2)}\, g_{rk}\ .$$

Substituting (4.7) into (4.3), we obtain

$$(4.8)\qquad \begin{aligned} &R_{irkj} + R_{rijk} + \frac{1}{2-n}(g_{ij}A_{rk} - g_{ik}A_{rj} - g_{rj}A_{ik} + g_{rk}A_{ij}) \\ &\qquad + \frac{B}{(n-1)(n-2)}(g_{ij}g_{kr} - g_{ik}g_{rj}) = \\ &= \overset{m}{R}_{irkj} + \overset{m}{R}_{rijk} + \frac{1}{2-n}(g_{ij}\overset{m}{A}_{rk} - g_{ik}\overset{m}{A}_{rj} - g_{rj}\overset{m}{A}_{ik} + g_{kr}\overset{m}{A}_{ij}) \\ &\qquad + \frac{\overset{m}{B}}{(n-1)(n-2)}(g_{ij}g_{kr} - g_{ik}g_{rj}). \end{aligned}$$

Taking into account (3.3), we obtain the following relations

$$\overset{m}{R}_{irkj} = \overset{m}{R}_{rijk}\ ;\quad \overset{m}{A}_{rk} = 2\overset{m}{R}_{rk},\quad \overset{m}{B} = 2\overset{m}{R}\ .$$

We can see that the tensor on the right-hand side of (4.8) is equal to $2C_{irkj}$ where C_{irkj} denotes the corresponding component of Weyl's conformal curvature tensor of the adjoint Riemannian space.

We have proved

THEOREM 2. *In $SW\text{-}O_n$ with $\gamma_k \neq 0$, the tensor on the left-hand side of (4.8) is equal to* $2C_{irkj}$.

COROLLARY 2. *If the adjoint Riemannian space is conformally Euclidean, the tensor on the left-hand side of (4.8) in $SW\text{-}O_n$ is equal to 0.*

REFERENCES

|1| T.Ōtsuki, *On general connection I, Math. J.Okayama Univ. 9(1959–1960) 99–164.*

|2| A.Moór, *Ōtsukische Übertragung mit rekurrenten Mass-tensor, Acta Sci. Math. Szeged 1978.*

|3| A.Moór, *Über die Weräderung der Länge der Vektoren in Weyl-Ōtsukischen Räumen, Acta Sci.Math. Szeged, 1979.*

|4| M.Prvanović, *Weyl-Ōtsuki Spaces of the Second and Third Kind, Zbornik radova Prirodno-matematičkog fakulteta u Novom Sadu,1982,349–*

|5| M.Prvanović, *On a Special Connection in an Ōtsuki Space, (to appear), Tensor.*

Received by the editors April 19,1983.

REZIME

WEYL-ŌTSUKI-JEVI PROSTORI DRUGE VRSTE SA SPECIJALNIM TENZOROM P

U ovom radu, definisani su Weyl-Ōtsuki-jevi prostori druge vrste reda n (u oznaci $SW\text{-}O_n$) sa specijalnim tenzorom P oblika $P^i_j = S(x)\delta^i_j$ gde je S(x) C^r funkcija (nenula i nelinearna). Ovo je prvi stepen uopštenja od Rimanovih prostora ka metričkim Ōtsuki-jevim prostorima sa regularnom opštom koneksijom. Ovde su uspostavljene veze izmedju tenzora krivine i tenzora konforne krivine u pridruženom Rimanovom prostoru i odgovarajućih objekata u $SW\text{-}O_n$.