Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 12 (1982)
Review of Research Faculty of Science-University of Novi Sad, Volume 12(1982)

ON BISYMMETRIC [n,m]-GROUPOIDS

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ABSTRACT

In this paper bisymmetric [n,m]-groupoids (defined in |6|) are considered. It is shown that every bisymmetric [n,m]-groupoid Q(f) is commutative, if the range of at least one of its component operations is Q. A corollary of the preceding proposition is that there do not exist proper bisymmetric [n,m]-quasigroups (which was proved in |6|). It is also shown that from the commutativity and mutual mediality of the component operations follows the bisymmetry of an [n,m]-groupoid. A characterization of the components which are n-quasigroups of a bisymmetric [n,m]-groupoid is given. Necessary and sufficient conditions for an [n,m]-groupoid to be bisymmetric with n-quasigroup components are obtained.

First we shall give some basic definitions. The notions from the general theory of n-quasigroups can be found in |1|.

Instead of $x_p, x_{p+1}, \dots, x_q$ we shall write $\{x_i\}_{i=p}^{q}$ or x_p^q. If $p > q$, then x_p^q will be considered empty.

An n-groupoid Q(f) is called commutative iff the following identity holds

$$f(x_1^n) = f(x_{\phi(1)}^{\phi(n)}) ,$$

for every permutation ϕ of the set $N_n = \{1, \dots, n\}$.

An n-groupoid Q(f) is totally symmetric iff

$$f(x_1^n) = x_{n+1} \quad \text{implies} \quad f(x_{\psi(1)}^{\psi(n)}) = x_{\psi(n+1)} \quad \text{for all} \quad x_i \in Q ,$$

$i = 1,\dots,n+1$ and every permutation ψ of N_{n+1}.

Two n-groupoids $Q(f)$ and $Q(g)$ are mutually medial iff

$$f(\{g(x_{i1}^{in})\}_{i=1}^{n}) = g(\{f(x_{1j}^{nj})\}_{j=1}^{n})$$

holds for all $x_{ij} \in Q$, $i,j = 1,\dots,n$. A n-groupoid which is medial to itself is called medial.

An n-groupoid $Q(f)$ is bisymmetric iff

$$f(\{f(x_{i1}^{in})\}_{i=1}^{n})$$

is invariant for every permutation of elements $x_{ij} \in Q$, $i,j = 1,\dots,n$.

Let Q be a nonempty set, n and m positive integers and $f: Q^n \to Q^m$. Then $Q(f)$ is called an $[n,m]$-groupoid. The n-ary operations $f_1,\dots,f_m$ defined by

$$f(x_1^n) = (y_1^m) \iff (\forall i \in N_m)\ y_i = f_i(x_1^n)$$

are called the component operations of f and this is denoted by $f = (f_1,\dots,f_m)$. An $[n,m]$-groupoid is proper iff n,m, $|Q| \geq 2$.

An $[n,m]$-groupoid $Q(f)$ is called an $[n,m]$-quasigroup (or multiquasigroup) (|2|,|3|) iff for every injection $\phi: N_n \to N_{n+m}$ and every $(a_1^n) \in Q^n$ there exists a unique $(b_1^{n+m}) \in Q^{n+m}$ such that

$$f(b_1^n) = (b_{n+1}^{n+m}) \quad \text{and} \quad b_{\phi(i)} = a_i,\ i = 1,\dots,n.$$

An $[n,m]$-groupoid $Q(f)$ is commutative iff

$$f(x_1^n) = f(x_{\phi(1)}^{\phi(n)})$$

holds for all $x_i \in Q$, $i = 1,\dots,n$ and every permutation ϕ of N_n. Obviously, an $[n,m]$-groupoid is commutative iff all its component operations are commutative n-groupoids.

Let $Q(f)$ be an $[n,m]$-groupoid, $f=(f_1,\dots,f_m)$. If $X = [x_{ij}]$ is an $n \times p$ array of elements from Q, then an $m \times p$ array $C_f(X) = [y_{ij}]$ is defined by $y_{ij} = f_i(x_{1j}^{nj})$, $i=1,\dots,m$, $j=1,\dots,p$.

If $X = [x_{ij}]$ is an $r \times n$ array of elements from Q, then an $r \times m$ array $R_f(X) = [z_{ij}]$ is defined by $z_{ij} = f_j(x_{i1}^{in})$, $i=1,\dots,r$, $j=1,\dots,m$. An [n,m]-groupoid Q(f) is said to be bisymmetric (|6|) iff for all $n \times n$ arrays $X = [x_{ij}]$ of elements from Q and every permutation ϕ of $N_n \times N_n$

$$C_f R_f(X) = C_f R_f(X^{\phi}) ,$$

where $X^{\phi} = [x_{\phi(i,j)}]$, holds.

It is obvious that the components of a bisymmetric [n,m]-groupoid are bisymmetric. But from the bisymmetry of the components the bisymmetry of an [n,m]-groupoid does not follow (|6|). In |6| it is also shown that there are bisymmetric [n,m]-groupoids which are not commutative. But this does not hold for a class of bisymmetric [n,m]-groupoids which are described in the following proposition.

If Q(f) is an n-groupoid, by Rf we shall denote the range of f.

THEOREM 1. *Let* Q(f) *be a bisymmetric* [n,m]*-groupoid,* $f=(f_1,\dots,f_m)$ *, such that there is a component operation* f_k *for which* $Rf_k = Q$. *Then each of the component operations* f_i, $i=1,\dots,m$, *is commutative.*

P r o o f. Q(f) is a bisymmetric [n,m]-groupoid, so for every every $\ell \in N_m$ and any permutation ϕ of $N_n \times N_n$, the following identity holds

$$f_\ell(\{f_k(x_{i1}^{in})\}_{i=1}^{n}) = f_\ell(\{f_k(x_{\phi(i,1)}^{\phi(i,n)})\}_{i=1}^{n}) .$$

If a_i, $i=1,\dots,n$ are arbitrary elements from Q, then, since $Rf_k = Q$, there exist elements $b_{ij} \in Q$, $i,j=1,\dots,n$ such that

$$f_k(b_{i1}^{in}) = a_i,\ i=1,\dots,n .$$

If ϕ is an arbitrary permutation of the set N_n and ψ a permutation of $N_n \times N_n$ such that $\psi(p,q) = (\phi(p),q)$ for all $p,q \in N_n$, then

$$f_\ell(a_1^n) = f_\ell(\{f_k(b_{\phi(i),1}^{\phi(i),n})\}_{i=1}^{n} = = f_\ell(a_{\phi(1)}^{\phi(n)}) .$$

Hence, f_ℓ is a commutative n-groupoid for all $\ell \in N_m$.

COROLLARY 1.1. *Every bisymmetric* $[n,m]$*-groupoid* $Q(f), f=(f_1,\dots,f_m)$ *, such that there is a component operation* f_k *for which* $Rf_k = Q$, *is a commutative* $[n,m]$*-groupoid.*

COROLLARY 1.2. *In* |6| *it is proved that every bisymmetric multiquasigroup is commutative (which implies that there are no proper bisymmetric multiquasigroups because in* |7| *it is proved that there are no proper commutative multiquasigrpups). This is a consequence of Theorem 1 since all components of an* $[n,m]$*-quasigroup* $Q(f)$, $f=(f_1,\dots,f_m)$, *are* n*-quasigroups, hence* $Rf_k = Q$ *for all* $k \in N_m$.

THEOREM 2. *Let* $Q(f)$ *be an* $[n,m]$*-groupoid,* $f=(f_1,\dots,f_m)$. *If all component operations* f_i, $i=1,\dots,m$ *are commutative and every two component operations are mutually medial, then* $Q(f)$ *is a bisymmetric* $[n,m]$*-groupoid.*

P r o o f. If f_k and f_ℓ are two commutative n-groupoids which are mutually medial, then it can be proved (by a similar argument as it is done in |5|, where the special case $f_\ell = f_k$ was considered) that

$$f_k(\{f_\ell(x_{i1}^{in})\}_{i=1}^{n}) = f_k(\{f_\ell(x_{\phi(i,1)}^{\phi(i,n)})\}_{i=1}^{n})$$

holds for all $x_{ij} \in Q$, $i,j=1,\dots,n$ and any permutation ϕ of $N_n \times N_n$. Hence, $Q(f)$ is a bisymmetric $[n,m]$-groupoid.

THEOREM 3. *Let* $Q(f)$ *be a bisymmetric* $[n,m]$*-groupoid,* $f=(f_1,\dots,f_m)$. *If* f_k *is an* n*- quasigroup, then there exist an Abelian group* $Q(+)$ *and an element* $a \in Q$ *such that*

$$f_k(x_1^n) = \alpha \sum_{i=1}^{n} x_i + a ,$$

where α *is an automorphism of the group* $+$.

P r o o f. Since $Q(f)$ is bisymmetric it follows that f_k is a bisymmetric n-quasigroup. This means that f_k is a medial n-quasigroup, hence there exist an Abelian group $Q(+)$ and an element $a \in Q$ such that

$$f_k(x_1^n) = \sum_{i=1}^{n} \alpha_i x_i + a$$

where α_i, $i=1,\dots,n$ are automorphisms of the group $+$ ($|1|$). From Theorem 1 it follows that f_k is commutative. If in $f_k(x_1^n) = f_k(x_{\phi(1)}^{\phi(n)})$ we put $x_1 = x_{\phi(2)} = x$ and 0 otherwise (where 0 is the neutral element of the group +), we get

$$\alpha_1 x + \alpha_2 0 + \dots + \alpha_n 0 + a = \alpha_1 0 + \alpha_2 x + \alpha_3 0 + \dots + \alpha_n 0 + a\ ,$$

thus $\alpha_1 x = \alpha_2 x$ for all $x \in Q$. By a similar argument it can be shown that $\alpha_1 = \dots = \alpha_n$.

THEOREM 4. *Let* $Q(f)$ *be an* $[n,m]$*-groupoid,* $f=(f_1,\dots,f_m)$. $Q(f)$ *is a bisymmetric* $[n,m]$*-groupoid the components of which are* n*-quasigroups iff there exist an Abelian group* $Q(+)$, *elements* $a_i \in Q$ $i=1,\dots,m$ *and automorphisms* α_i, $i=1,\dots,m$ *of the group* + *such that*

$$f_i(x_1^n) = \alpha_i \sum_{j=1}^{n} x_j + a_i,\ i=1,\dots,m.$$

P r o o f. Let $Q(f)$ be a bisymmetric $[n,m]$-groupoid and let all component operations f_i, $i=1,\dots,m$ be n-quasigroups. Then by Theorem 3 it follows that there exist Abelian groups $Q(\overset{i}{+})$, $i=1,\dots,m$, and elements $a_i \in Q$, $i=1,\dots,m$ such taht

$$f_i(x_1^n) = \alpha_i(x_1 \overset{i}{+} \dots \overset{i}{+} x_n) \overset{i}{+} a_i,\ i=1,\dots,m\ ,$$

where α_i is an automorphism of the group $\overset{i}{+}$, $i=1,\dots,m$.

Let k,j be arbitrary elements from N_m. The identity

$$f_k(\{f_j(x_{i1}^{in})\}_{i=1}^{n}) = f_k(\{f_j(x_{\phi(i,1)}^{\phi(i,n)})\}_{i=1}^{n}),$$

where ϕ is a permutation of $N_n \times N_n$, then, can be written in th form

$$\alpha_k((\alpha_j(x_{11} \overset{j}{+} \dots \overset{j}{+} x_{1n}) \overset{j}{+} a_j) \overset{k}{+} \dots \overset{k}{+} (\alpha_j(x_{n1} \overset{j}{+} \dots \overset{j}{+} x_{nn}) \overset{j}{+} a_j)) \overset{k}{+} a_k =$$

$$= \alpha_k((\alpha_j(x_{\phi(1,1)} \overset{j}{+} \dots \overset{j}{+} x_{\phi(1,n)}) \overset{j}{+} a_j) \overset{k}{+} \dots \overset{k}{+} (\alpha_j(x_{\phi(n,1)} \overset{j}{+} \dots$$

$$\dots \overset{j}{+} x_{\phi(n,n)}) \overset{j}{+} a_j) \overset{k}{+} a_k\ .$$

Let e_i be the neutral element of the group $\overset{i}{+}$, $i=1,\dots,m$. If in the preceding equality we put $x_{11}=x_{\phi(1,1)}=x$, $x_{12}=x_{\phi(2,1)}=y$, $x_{in}=x_{\phi(i,n)}=\alpha_j^{-1}(\overset{j}{-}a)$, $i=1,\dots,n$; $x_{i2}=\alpha_j^{-1}e_k$, $i=2,\dots,n$; $x_{\phi(i,1)}=\alpha_j^{-1}e_k$, $i=3,\dots,n$; $x_{\phi(n,n-1)}=\alpha_j^{-1}e_k$ and $x_{i\ell}=x_{\phi(i,\ell)}=e_j$ otherwise, then we get

$$(\alpha_j x\overset{j}{+}\alpha_j y)\overset{k}{+}e_k\overset{k}{+}\dots\overset{k}{+}e_k=(\alpha_j x)\overset{k}{+}(\alpha_j y)\overset{k}{+}e_k\overset{k}{+}\dots\overset{k}{+}e_k\overset{k}{+}(e_k\overset{j}{+}e_k)\ ,$$

thus

$$\alpha_j x\overset{j}{+}\alpha_j y=(\alpha_j x)\overset{k}{+}(\alpha_j y)\overset{k}{+}(e_k\overset{j}{+}e_k)\ .$$

So, for all $x,y\in Q$

$$x\overset{j}{+}y=x\overset{k}{+}y\overset{k}{+}f,$$

where $f=e_k\overset{j}{+}e_k$, which yields the identity

$$f_j(x_1^n)=\alpha_j(x_1\overset{k}{+}\dots\overset{k}{+}x_n)\overset{k}{+}\bar{a}_j$$

where $\bar{a}_j=nf\overset{k}{+}a_j$.

The converse part of the theorem can be proved by a straightforward computation.

The preceding theorem is a generalization of Proposition 3.1. from |6|. In fact, in |6| a WTSB [n,m]-groupoids are defined (as a generalization of C^{n+1}-systems introduced in |4|). An [n,m]-groupoid is WTSB iff it is bisymmetric and all its component operations are totally symmetric n-quasigroups. If $Q(f)$ is a WTSB [n,m]-groupoid, then from Theorem 4 it follows that

$$f_i(x_1^n)=\alpha_i\sum_{j=1}^{n}x_j+a_i,\quad i=1,\dots,m.$$

If $f_i(x_1^n)=y$, then putting in the last equality $x_2=\dots=x_n=0$, because of the total symmetry of f_i, we get

$$\alpha_i x_1+\alpha_i y+a_i=0,\quad\text{and}\quad \alpha_i y+a_i=x_1\ ,$$

which implies $\alpha_i x_1=-x_1$ for all $x_1\in Q$.

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Received by the editors May 10,1983.

REZIME

O BISIMETRIČNIM [n,m]-GRUPOIDIMA

U ovom radu razmatrani su bisimetrični [n,m]-grupoidi. Pokazano je da je svaki bisimetričan [n,m]-grupoid Q(f) koji ima bar jednu komponentnu operaciju čiji je kodomen Q, komutativan. Posledica ovog tvrdjenja je da ne postoji prava bisimetrična [n,m]-kvazigrupa (što je dokazano u |6|). Takodje je dokazano da iz komutativnosti i uzajamne medijalnosti komponentnih operacija sledi bisimetričnost [n,m]-grupoida. Data je jedna karakterizacija komponenata bisimetričnog [n,m]-grupoida koje su n-kvazigrupe. Navedeni su potrebni i dovoljni uslovi da [n,m]-grupoid bude bisimetričan sa komponentama koje su n-kvazigrupe.