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# CYCLIC n-QUASIGROUPS

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ABSTRACT

In this paper cyclic n-groupoids (Definition 1), i.e. cyclic n-quasigroups (because every cyclic n-groupoid is necessarily a cyclic n-quasigroup) are considered. Different equivalent definitions of a cyclic n-groupoid are given. Examples of cyclic n-quasigroups are listed. Circular parastrophes of an n-quasigroup, which are suitable for the study of cyclic n-quasigroups, are defined. It is determined which parastrophes of a cyclic n-quasigroup are cyclic. It is shown that an n-quasigroup which is isotopic to a cyclic n-quasigroup must be isotopic to all its circular parastrophes and conditions under which its parastrophes are isotopic to a cyclic n-quasigroup are given. Some consequences which follow from the assumption that an n-quasigroup is isotopic to one of its parastrophes are obtained. Using these consequences a theorem which gives necessary and sufficient conditions for an n-quasigroup to be isotopic to a cyclic n-quasigroup is proved.

First we give some basic definitions and notations. Other notions from the theory of n-ary quasigroups can be found in [1] .

The sequence $x_m, x_{m+1}, \dots, x_n$ we denote by $\{x_i\}_{i=m}^{n}$ or $x_m^n$. If $m > n$, then $x_m^n$ will be considered empty. The sequence $x, x, \dots, x$ (n times) will be denoted by $\overset{n}{x}$ .

An n-groupoid (Q,A) is called an n-quasigroup iff the equation $A(a_1^{i-1},x,a_{i+1}^n)=b$ has a unique solution x for every $a_1^n, b\in Q$ and every $i\in N_n=\{1,\dots,n\}$.

An n-quasigroup (Q,A) is isotopic to n-quasigroup (Q,B) iff there exists a sequence $T=(\alpha_1^{n+1})$ of permutations of Q such that the following identity

$$B(x_1^n) = \alpha_{n+1}^{-1}A(\{\alpha_i x_i\}_{i=1}^n)$$

holds. T is called an isotopism, and by $A^T=B$ we denote that A is isotopic to B by T. $T^{-1}$ is defined by $T^{-1}=(\{\alpha_i^{-1}\}_{i=1}^{n+1})$.

If (Q,A) is an n-quasigroup and $\sigma\in S_{n+1}$, where $S_{n+1}$ is the symmetric group of degree n+1, then the n-quasigroup $A^\sigma$ defined by

$$A^\sigma(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \iff A(x_1^n) = x_{n+1} ,$$

is called a σ-parastrophe (or simply parastrophe) of A. If $\sigma,\tau\in S_{n+1}$ then $(A^\sigma)^\tau = A^{\sigma\tau}$ and

$$A(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \iff A^\tau(x_{\sigma\tau(1)}^{\sigma\tau(n)}) = x_{\sigma\tau(n+1)} .$$

If $T=(\alpha_1^{n+1})$ is an isotopy of A, then $(A^T)^\sigma = (A^\sigma)^{T^\sigma}$, where $T^\sigma = (\alpha_{\sigma 1}^{\sigma(n+1)})$.

An n-quasigroup (Q,A) is called idempotent iff for every $x\in Q$ $A(\overset{n}{x}) = x$.

An n-quasigroup (Q,A) is called totally symmetric iff A coincides with all its parastrophes.

DEFINITION 1. *An* n-*groupoid* (Q,A) *is called cyclic iff the following identity holds*

$$A(A(x_1^n),\ x_1^{n-1}) = x_n .$$

A binary groupoid or quasigroup (Q,·) is called semi-symmetric iff it satisfies the identity $(xy)x=y$. So, cyclic n-groupoids defined in Definition 1 are a generalization of semi-symmetric groupoids. Semi-symmetric quasigroups were investigated in various directions. Some results on semi-symmetric quasigroups (mostly obtained by A. Sade) are used in the enumeration and classification of latin squares and some classes of semi-symmetric quasigroups are related to geometry of plane curves. There is also a connection between indempotent semi-symmetric quasigroups and balanced incomplete block designs ([3]) and some other combinatorial designs ([2]).

The definition of a cyclic n-groupoid can be also given in another form. The next definition is equivalent to Definition 1.

DEFINITION 1'. *An* n-*groupoid* (Q,A) *is cyclic iff for all* $x_1^{n+1} \in Q$

$$A(x_1^n) = x_{n+1} \Longrightarrow A(x_{n+1}, x_1^{n-1}) = x_n .$$

Using the preceding implication it is easy to obtain the following definition equivalent to the preceding ones.

DEFINITION 1". *An* n-*groupoid* (Q,A) *is cyclic iff for every* $i \in N_n$ *and all* $x_1^{n+1} \in Q$

$$A(x_1^n) = x_{n+1} \Longleftrightarrow A(x_{i+1}^{n+1}, x_1^{i-1}) = x_i .$$

If (Q,A) is a cyclic n-groupoid, then for all $i \in N_n$ and all $a_1^{i-1}$, $a_{i+1}^{n+1} \in Q$, the equation

$$(1) \qquad A(a_1^{i-1}, x, a_{i+1}^n) = a_{n+1}$$

is by Definition 1" equivalent to

$$A(a_{i+1}^{n+1}, a_1^{i-1}) = x .$$

This means that the equation (1) has a unique solution for x, hence the following proposition is valid.

PROPOSITION 1. *Every cyclic* n-*groupoid is an* n-*quasigroup.*

We now give some examples of cyclic n-quasigroups.

1. Let (Q,+) be an arbitrary Abelian group (binary), $n \geq 2$, a an arbitrary element from Q, and $\phi$ an automorphism of

the group $(Q,+)$ such that $\phi a = -a$, and if n is even then for all $x \in Q$, $\phi^{n+1}x = -x$, and if n is odd then $\phi^{n+1}$ is the identity mapping. Then by

$$(2) \qquad A(x_1^n) = \phi x_1 - \phi^2 x_2 + \phi^3 x_3 - \ldots + (-1)^{n+1}\phi^n x_n + a$$

a cyclic n-quasigroup $(Q,A)$ is defined. Among cyclic n-quasigroups defined by (2) there are n-quasigroups which are neither totally symmetric nor idempotent and which are not n-groups.

An automorphism of an arbitrary Abelian group, which satisfies the given conditions is given by $\phi : x \mapsto -x$, and the cyclic n-quasigroup obtained that way for n even need not be idempotent nor n-group, but it is always totally symmetric. If n is odd, $\phi$ the identity mapping and $a = 0$, a cyclic n-quasigroup which is not totally symmetric can be obtained.

In an arbitrary ring with unity $(R,+,.)$ the mapping $\phi : x \mapsto bx$, where b is an invertible element, is an automorphism of the additive group $(R,+)$. If b is such that for n even $b^{n+1} = -1$, and for n odd $b^{n+1} = 1$, then by (2) a cyclic n-quasigroup $(R,A)$ is defined.

If R is the ring $Z_q$ of integers modulo q, then, for example, for $q = 15$, $b = 4$ and an arbitrary odd n a cyclic n-quasigroup of order 15 is obtained, for $q \in \{4,61,122\}$, $b = 3$, $n = 4$ a cyclic 4-quasigroup of order q is obtained etc.

2. Let $(Q,\cdot)$ be an arbitrary group, $\alpha$ an automorphism of this group such that $\alpha^{n+1}$ is the identity mapping, a an element of the center of the given group such that $\alpha a = a$. Then by

$$A(x_1^n) = \alpha x_1^{-1} \alpha^2 x_2^{-1} \ldots \alpha^n x_n^{-1} a$$

a cyclic n-quasigroup is defined.

Now we shall define one kind of parastrophes which are suitable for the study of cyclic n-quasigroups.

DEFINITION 2. *If* $(Q,A)$ *is an* n-*quasigroup*, $i \in N_n \cup \{0\}$ *then each of the* n+1 *parastrophes* $A^{\sigma_i}$, $i = 1, \ldots, n+1$, *defined by* $\sigma_i : a \mapsto a+i \pmod{(n+1)}$ *is called circular parastrophe of the* n-*quasigroup* $(Q,A)$.

From the definitions 1´, 1" and 2 the following propositions can be obtained.

THEOREM 1. *An* n-*quasigroup* (Q,A) *is cyclic iff* A *coicides with its parastrophe* $A^{\sigma_i}$, *where* $\sigma_i$ *is a circular parastrophe such that* i *is relatively prime to* n+1.

THEOREM 2. *An* n-*quasigroup* (Q,A) *is cyclic iff it coincides with all its circular parastrophes.*

In the binary case ([4]) all parastrophes of a cyclic quasigroup are cyclic. But, for an n-quasigroup, $n > 2$, this is not the case. Let (Q,A) be a cyclic n-quasigroup and $A^\sigma$ its parastrophe. For all $x_1^{n+1} \in Q$ and every $i \in N_n$ we have

$$A^\sigma(x_1^n) = x_{n+1} \iff A(x_{\sigma^{-1}1}^{\sigma^{-1}n}) = x_{\sigma^{-1}(n+1)} \iff A^{\sigma_i}(x_{\sigma^{-1}\sigma_i 1}^{\sigma^{-1}\sigma_i n}) =$$

$$= x_{\sigma^{-1}\sigma_i(n+1)} \iff A(x_{\sigma^{-1}(1+i)}^{\sigma^{-1}(n+i)}) = x_{\sigma^{-1}(n+1+i)} \iff$$

$$\iff A^\sigma(x_{\sigma^{-1}(\sigma(1)+i)}^{\sigma^{-1}(\sigma(n)+i)}) = x_{\sigma^{-1}(\sigma(n+1)+i)}$$

where all indices are reduced modulo n+1. Hence, if $\sigma$ is such that for an $i \in N_n$ and all $t \in N_n$

$$\sigma^{-1}(\sigma(t)+i) = t+1 \pmod{(n+1)} ,$$

i.e.

$$\sigma(t+1) = \sigma(t) + i \pmod{(n+1)} , \tag{3}$$

then the parastrophe $A^\sigma$ is cyclic. So we have proved

THEOREM 3. *Let* (Q,A) *be a cyclic* n-*quasigroup and* $A^\sigma$ *its parastrophe such that* $\sigma$ *satisfies (3). Then* $A^\sigma$ *is a cyclic* n-*quasigroup.*

Of course, the circular parastrophes always satisfy (3), but there are other noncircular parastrophes which satisfy (3). Other parastrophes of a cyclic n-quasigroup which do not satisfy the condition (3) need not be cyclic. If n=2 then (3) is satisfied for all permutations $\sigma \in S_3$.

We shall now consider isotopes of cyclic n-quasigroups.

THEOREM 4. *If an* n*-quasigroup* A *is isotopic to a cyclic* n*-quasigroup* B, *then* A *is isotopic to each of its circular parastrophes* $A^{\sigma^i}$. *Then all parastrophes of* A *which satisfy (3) are isotopes of cyclic* n*-quasigroups.*

P r o o f. Since B is a cyclic n-quasigroup, it coincides with all its circular parastrophes. Corresponding parastrophes of isotopic n-quasigroups are isotopic, hence, the circular parastrophes of A are isotopic to the corresponding parastrophes of B, which means that all circular parastrophes of A are isotopic to B. Since the isotopism is transitive, it follows that A is isotopic to each of its circular parastrophes.

The parastrophes of the cyclic n-quasigroup B which satisfy (3) are cyclic n-quasigroups, hence, the corresponding parastrophes of A are isotopes of cyclic n-quasigroups.

THEOREM 5. *Let* (Q,A) *be an* n*-quasigroup isotopic to its parastrophe* $A^\sigma$ *by an isotopism* $T = (\alpha_1^{n+1})$, $A^T = A^\sigma$, *where* $\sigma \in S_{n+1}$ *is an arbitrary cycle of length* n+1. *Then there exist a permutation* $\theta \in S_Q$ *and an* n*-quasigroup* (Q,B) *which is isotopic to* A, *such that* B *is isotopic to* $B^\sigma$ *by the isotopism* $(1,\ldots,1,\theta^{-1}\alpha_{n+1}\alpha_{\sigma(n+1)}\ldots\alpha_{\sigma^n(n+1)}\theta)$.

P r o o f. Let B be an arbitrary isotope of A, $B = A^S$, $S = (\beta_1^{n+1})$. Since $A^T = A^\sigma$, it follows

$$(((B^{S^{-1}})^T)^{\sigma^{-1}})^S = B.$$

For every n-quasigroup (Q,C) we have $(C^{\sigma^{-1}})^S = (C^{S^\sigma})^{\sigma^{-1}}$, hence,

$$((B^{S^{-1}T})^{S^\sigma})^{\sigma^{-1}} = B,$$

and

$$B^{S^{-1}TS^\sigma} = B^\sigma,$$

which means that every isotope B of the n-quasigroup A is isotopic its parastrophe $B^\sigma$.

Now we show that there exists an isotope B of A, such that there is an isotopism of B to $B^\sigma$ which has the form described in the theorem. Since $S^{-1}TS^\sigma = (\{\beta_i^{-1}\alpha_i\beta_{\sigma i}\}_{i=1}^{n+1})$, then

$B^{(\{\beta_i^{-1}\alpha_i\beta_{\sigma i}\}_{i=1}^{n+1})} = B$. If we put

$$\beta_1^{-1}\alpha_1\beta_{\sigma 1} = 1, \dots, \beta_n^{-1}\alpha_n\beta_{\sigma n} = 1,$$

then

$$\alpha_1\beta_{\sigma 1} = \beta_1, \dots, \alpha_n\beta_{\sigma n} = \beta_n .$$

If we take $\beta_{n+1}$ to be an arbitrary permutation from Q, then from the i-th of the preceding equations, where $i=\sigma^{-1}(n+1)$, we get $\beta_i$, then from the j-th equation, where $j=\sigma^{-1}i=\sigma^{-2}(n+1)$, we get $\beta_j$ and so on, at the end we get $\beta_{\sigma^{-n}(n+1)}$. Since $\sigma$ is a cycle of length n+1 we have obtained all permutations $\beta_1,\dots,\beta_n$:

$$(4)\qquad \begin{aligned} \beta_{\sigma^{-1}(n+1)} &= \alpha_{\sigma^{-1}(n+1)}\beta_{n+1} , \\ \beta_{\sigma^{-2}(n+1)} &= \alpha_{\sigma^{-2}(n+1)}\beta_{\sigma^{-1}(n+1)} = \alpha_{\sigma^{-2}(n+1)}\alpha_{\sigma^{-1}(n+1)}\beta_{n+1} \\ &\dots\dots\dots\dots\dots\dots \\ \beta_{\sigma^{-n}(n+1)} &= \alpha_{\sigma^{-n}(n+1)}\alpha_{\sigma^{-n+1}(n+1)}\dots\alpha_{\sigma^{-1}(n+1)}\beta_{n+1} . \end{aligned}$$

Then the (n+1)-th component of the isotopism $S^{-1}TS^{\sigma}$ has the form:

$$\beta_{n+1}^{-1}\alpha_{n+1}\beta_{\sigma(n+1)} = \beta_{n+1}^{-1}\alpha_{n+1}\beta_{\sigma^{-n}(n+1)} =$$

$$= \beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma^{-n}(n+1)}\beta_{\sigma^{-n+1}(n+1)} =$$

$$= \dots = \beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma^{-n}(n+1)}\alpha_{\sigma^{-n+1}(n+1)}\dots\alpha_{\sigma^{-1}(n+1)}\beta_{n+1} =$$

$$= \beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma(n+1)}\alpha_{\sigma^2(n+1)}\dots\alpha_{\sigma^n(n+1)}\beta_{n+1} .$$

So, we have obtained that for an arbitrary $b_{n+1} \in S_Q$ the isotope B of A by the isotopism $S=(\beta_1^{n+1})$, where $\beta_i$, i=1,...,n are given by (4), $B=A^S$, is an n-quasigroup which is isotopic to its parastrophe $B^{\sigma}$ by the isotopism $(1, \dots, 1, \beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma(n+1)}\dots\alpha_{\sigma^n(n+1)}\beta_{n+1})$ .

REMARK. A special case of this theorem, when n=2, is a generalization of a theorem from [3], where the correspondi-ng proposition is proved for only one parastrophe $A^{\sigma}$, $\sigma \in S_3$.

THEOREM 6. *An* n-*quasigroup* (Q,A) *is isotopic to a cyclic* n-*quasigroup iff there exists a circular parastrophe* $A^{\sigma_i}$, *where* i *is relatively prime to* n+1, *such that* A *is isotopic to* $A^{\sigma_i}$ *by an isotopism* $T=(\alpha_1^{n+1})$ *where* $\alpha_{n+1}\alpha_{\sigma_i(n+1)}\cdots\alpha_{\sigma_i^n(n+1)}=1$.

P r o o f. If there is a parastrophe $A^{\sigma_i}$ which satisfies the conditions given in the theorem, then by the preceding theorem A is isotopic to an n-quasigroup B which coincides with its parastrophe $B^{\sigma_i}$. If B coincides with $B^{\sigma_i}$, then by Theorem 1 B is cyclic, hence, A is isotopic to a cyclic n-quasigroup.

Conversely, let A be isotopic to a cyclic n-quasigroup B, $A^S=B$, $S=(\beta_1^{n+1})$. Let $A^{\sigma_i}$ be a circular parastrophe such that i is relatively prime to n+1. Then $\sigma_i$ is a cycle of lenght n+1 and $(A^{\sigma_i})^{S^{\sigma_i}}=B^{\sigma_i}$. Since B is cyclic $B=B^{\sigma_i}$. From $A^S=B$, $(A^{\sigma_i})^{S^{\sigma_i}}=B$, it follows

$$A^{S(S^{\sigma_i})^{-1}}=A^{\sigma_i},$$

so, $S(S^{\sigma_i})^{-1}$ is an isotopism of A to $A^{\sigma_i}$. If we denote this isotopism by $T=S(S^{\sigma_i})^{-1}=(\alpha_1^{n+1})$, we shall have $S^{-1}TS^{\sigma_i}=I=$ $=(1,\ldots,1)$, i.e.

$$\beta_k^{-1}\alpha_k\beta_{\sigma_i k}=1,\quad k=1,\ldots,n+1 .$$

From these equalities, analogously as it is done in the proof of the preceding theorem, we get

$$\beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma_i(n+1)}\cdots\alpha_{\sigma_i^n(n+1)}\beta_{n+1}=1,$$

i.e.

$$\alpha_{n+1}\alpha_{\sigma_i(n+1)}\cdots\alpha_{\sigma_i^n(n+1)}=1.$$

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REZIME

### CIKLIČKE n-KVAZIGRUPE

U ovom radu razmotreni su ciklički n-grupoidi (definicija 1), odnosno n-kvazigrupe (jer je svaki ciklički n-grupoid n-kvazigrupa). Date su različite ekvivalentne definicije cikličkih n-kvazigrupa. Navedeni su primeri cikličkih n-kvazigrupa, medju njima i primeri n-kvazigrupa koje nisu idempotentne, ni totalno simetrične. Definisani su kružni parastrofi koji su pogodni za izučavanje cikličkih n-kvazigrupa. Utvrdjeno je koji parastrofi cikličke n-kvazigrupe moraju biti ciklički. Pokazano je da je n-kvazigrupa koja je izotopna cikličkoj n-kvazigrupi izotopna svim svojim kružnim parastrofima i navedeni su uslovi pod kojima su njeni parastrofi izotopni cikličkim n-kvazigrupama. Iz pretpostavke da je n-kvazigrupa izotopna svom parastrofu odredjenog tipa izvedene su razne posledice. Na osnovu tih posledica dokazana je teorema koja daje potrebne i dovoljne uslove pod kojima je n-kvazigrupa izotopna cikličkoj n-kvazigrupi.