Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu knjiga 12 (1982)
Review of Research Faculty of Science-University of Novi Sad, Volume 12(1982)

SEMIGROUPS WITH COMPLETELY SIMPLE KERNEL

Stojan Bogdanović and Silvija Gilezan
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

ABSTRACT

In the present paper some new characterizations of semigroups with a completely simple kernel by bi-ideals and left ideals are given. Also, in this paper we consider semigroups in which every proper left ideal (bi-ideal) is a completely simple semigroup and the isomorphism theorem of this semigroup is given.

INTRODUCTION

Semigroups which contain minimal ideals are considered by A.H. Clifford, |6|. If a semigroup contains at least one minimal left and at least one minimal right ideal, then it contains a completely simple kernel, or equivalently, it contains a quasi - ideal which is a group (see Theorem 3.3. |6| and Theorem 5.14. |18|). The structural theorem for this class of semigroups is is given by A.H.Clifford, |7| (see also Theorem 2.1. |13| and Theorem 1.1. |16|). The description of this class of semigroups by bi-ideals and AB-ideals is given by S.Bogdanović, |1|.

Semigroups in which every proper subsemigroup is a group are considered by G.Pollák and L.Rédei, |15|. Š.Schwarz, |17| worked on semigroups in which every proper left ideal is a group. The same class of semigroups was studied by R.Hrmová, |11| G.Čupona, |8| considered first a semigroup S in which every subset $Sx \neq S$ is a group, later, semigroups in which some left ideal is a group, |9|. This result is a special case of the result given by A.Clifford, |7|. Clearly, this class of semigroups is is more extensive that the one considered by Schwarz and Hrmová. The result of Ǵ.Čupona was an impulse for S.Milić and V.Pavlović, |13|, to give a structural description of semigroups in which some ideal is a completely simple semigroup, i.e. a semigroup with completely simple kernel. It is given an extension of a Rees matrix semigroup over a group in their paper. (See, also, |7| by A.H.Clifford). In |16| P.Protić and S.Bogdanović give the description of an extension of Rees matrix semigroup over a monoid.

Here will be given some new characterizations of semigropus with completely simple kernel by bi-ideals and left ideals (Theorem 2.1.). Also, in this paper we shall consider semigroups in which every proper left ideal (bi-ideal) is a completely simple semigroup (Theorem 2.1. and Theorem 6.1.), and next, semigroups in which every proper left ideal is a right group. In section 8. some isomorphism theorems will be given. Semigropus in which every left ideal is a left group are described by Bogdanović, |2|. Various generalizations of the results of Pollák-Rédei, |15| are given in |3| and |4|. Among other results, the description of semigroups in which every proper subsemigroup is simple is given in |4| (Theorem 2.1.).

2. SEMIGROUPS WITH COMPLETELY SIMPLE KERNEL

Here will be given some new characterizations of semigroups with completely simple kernel. Let us denote $M(G;I,J;P)$ a Rees matrix semigroup over a group G.

THEOREM 2.1. *The following conditions on a semigroup* S *are equivalent:*

1° *Some left ideal of* S *is a completely simple semigroup;*
2° *Some bi-ideal of* S *is a completely simple semigroup;*
3° *Some left ideal of* S *is a left group;*
4° S *has a completely simple kernel.*

P r o o f. $1^{\circ} \Rightarrow 2^{\circ}$. Let L be a left ideal of S which is a completely simple subsemigroup of S. Then $L = M(G;I,J;P)$, so the group $H_{ij} = \{(g;i,j) : g \in G\}$, $(i \in I,\ j \in J)$

is a bi-ideal of L. It follows that

$$H_{ij}SH_{ij} = H_{ij}SH_{ij}H_{ij} \subseteq H_{ij}SLH_{ij} \subseteq H_{ij}LH_{ij} \subseteq H_{ij} .$$

Therefore, H_{ij} is a bi-ideal of S.

$2^{o} \Rightarrow 4^{o}$ Let a bi-ideal B of S be a completely simple semigroup. Then a maximal subgroup G of B is a bi-ideal of B, so it is a bi-ideal of S. Consequently, by Theorem 1. |1|, S has a completely simple kernel.

$3^{o} \Rightarrow 4^{o}$. If a left ideal L of S is a left group, then L contains a right ideal G which is a group; G is a bi-ideal of S, so by Theorem 1. |1|, S has a completely simple kernel.

$4^{o} \Rightarrow 3^{o}$. If S contains a completely simple kernel K, then we can put $K = M(G;I,J;P)$, thus $L_j = \{(g;i,j) : g \in G\}$ $i \in I$, $j \in J$ is a left ideal of K. Semigroup L_j is a left group, so $L_j = L_j^2$, $j \in J$. It follows that $SL_j = SL_j^2 \subseteq SKL_j \subseteq KL_j \subseteq L_j$. Therefore, S has a left ideal which is a left group.

$4^{o} \Rightarrow 1^{o}$. It follows immediately.

Also, let us consider the following conditions:

5^{o} Some bi-ideal of S is a group;

6^{o} Some quasi-ideal of S is a group;

7^{o} S has at least one minimal left and at least one minimal right ideal.

According to Theorem 1. |1|, Theorem 3.2. |6| and Theorem 2.1. we get the following result.

THEOREM 2.2. *Conditions* 1^{o}-7^{o} *are equivalent on a semigroup* S .

3. SEMIGROUPS IN WHICH EVERY PROPER LEFT IDEAL A COMPLETELY SIMPLE SEMIGROUP

LEMMA 3.1. |2|. *Let* I *be a (left) two sided ideal of a semigroup* S, *and let* K *be a (left) simple subsemigroup of* S *such that* $K \cap I \neq \emptyset$, *then* $K \subseteq I$.

LWMMA 3.2. |2| *Let* I *be a proper two-sided ideal of* S *which is not a proper subset of any proper left ideal of* S . *Then*

a) $S \setminus I$ *is a left simple semigroup*

or

b) $S \setminus I = \{a\}$, $a^2 \in I$.

LEMMA 3.3. *A left ideal* L *of a completely simple semigroup* S *is a completely simple subsemigroup of* S.

P r o o f. It is known that S is the union of groups, so for each $a \in L$ there exists $a^{-1} \in S$ such that $a = aa^{-1}a$, $aa^{-1} = a^{-1}a$, $a^{-1} = a^{-1}aa^{-1} = a^{-1}a^{-1}a \in SL \subseteq L$ Thus L is the union of groups and because of the fact that each idempotent of S is a primitive idempotent, we have that L is a completely simple semigroup, (|10|, IV Theorem 2.4.).

LEMMA 3.4. *Every proper left ideal of a semigroup* S *is a completely simple semigroup if and only if* S *has a kernel* K *and one of the following condition hold:*

1° $S = K \cong M(G;I,J;P)$;

2° $K \cong M(G;I,J;P)$, $S \setminus K$ *is a left simple subsemigroup of* S *and for each* $a \in S \setminus K$, $Ka \supseteq K$;

3° K *is a left group*, $S \setminus K = \{a\}$ *and* $a^2 \in K$.

P r o o f. Let every proper left ideal of S be a completely simple semigroup. Then by Theorem 2.1. S has a kernel K which is a completely simple semigroup.

Assume that $K \neq S$. If L is a proper left ideal of S, then by hypothesis it is a completely simple semigroup. Obviously $K \cap L \neq \emptyset$ and by Lemma 3.1. $L \subseteq K$. Thus, K is the unique maximal left ideal of S. According to Lemma 3.2. $S \setminus K$ is a left simple subsemigroup of S or $S \setminus K = \{a\}$, $a^2 \in K$.

<u>The case:</u> $S \setminus K = T$ is a left simple semigroup. Let us take $a \in T$, then the left ideal $[a]_L$ generated by a is

$$[a]_L = a \cup Sa = a \cup (K \cup T)a = a \cup Ka \cup Ta = T \cup Ka$$

(because T is a left simple semigroup). If $[a]_L$ is a proper left ideal of S, then it is a completely simple semigroup and since $K \cap [a]_L \neq \emptyset$, by lemma 3.1. we have that $[a]_L \subset K$, i.e. $T \subset K$, which is impossible. Therefore $[a]_L = S$. It follows from this that $T \cup Ka = T \cup K$ and since $K \cap T = \emptyset$ we have that $K \subseteq Ka$ for each $a \in T$.

<u>The case:</u> $S \setminus K = \{a\}$, $a^2 \in K$. Let us assume that K is not a left group. Then there exists a proper left ideal L_j of K which is a left group and $a^2 \in L_j$. The left ideal generated by a is

$$[a]_L = a \cup Sa = a \cup (K \cup \{a\})a = a \cup a^2 \cup Ka .$$

Then $[a]_L^2 = a^2 \cup a^3 \cup a^4 \cup Ka$, thus from $[a]_L = [a]_L^2$ it follows that $a \in [a]_L^2 \subset K$, which is impossible. Therefore $[a]_L^2 \neq [a]_L$, so $[a]_L$ is not a completely simple semigroup, and $[a]_L = S$, i.e. $a \cup a^2 \cup Ka = K \cup a$. Immediately it follows that $a^2 \cup Ka \supseteq K$, thus $a^3 \cup Ka^2 \supseteq Ka$, and $K \subseteq a^2 \cup a^3 \cup Ka^2 \subset L_j$, i.e. $K \subseteq L_j$. This means that K is a left group. Contradiction. So we have proved that $S = K \cup \{a\}$, $a^2 \in K$ and K is a left group.

Conversely, if 1° holds then the proposition follows by Lemma 3.3. If 3° holds, let us take the proper left ideal L of S. Then $K \cap L \neq \emptyset$, thus $K \subseteq L$. If $K \neq L$, then $a \in L$, i.e. $S = L$, but this is impossible. Hence, $K = L$. The case 2°. If L is a proper left ideal of S and $L \subseteq K$, then the assertion follows by Lemma 3.3. If $L \not\subseteq K$, then $L \cap T \neq \emptyset$, so by Lemma 3.1. we have that $T \subset L$. For $a \in T$ is $K \subseteq Ka \subset KL \subset L$. Hence, $S = K \cup T = L$, i.e. $S = L$, which is not possible.

REMARK. Lemma 3.4. can be given in the following way: Every proper left ideal of a semigroup S is a completely simple semigroup if and only if S has a kernel $K \cong M(G;I,J;P)$ and $S \setminus K = \emptyset$ or K is the unique maximal left ideal of S. (This can be seen easy from the proof of Lemma 3.4.).

EXAMPLE. The semigroup S given by

	1	2	3	4	a
1	1	2	1	2	1
2	1	2	1	2	1
3	3	4	3	4	3
4	3	4	3	4	3
a	1	2	1	2	1

has a kernel $K = \{1,2,3,4\}$ which is a rectangular band, $S \setminus K = \{a\}$, $a^2 = 1 \in K$. But S has another maximal left ideal. It is $\{1,3,a\}$ and it is not a completely simple semigroup.

Let $K = M(G;I,J;P)$ be a Rees matrix semigroup and let T be a left simple semigroup such that $K \cap T = \emptyset$.

Let $\xi : p \to \xi_p$ be a mapping of T into the semigroup $T(I)$ of all mappings of I into itself, and $\eta : p \to \eta_p$ the mapping of T into the semigroup of all sirjections of J onto J, and for $p,q \in T$ let

(i) $$\xi_{pq} = \xi_q \xi_p , \quad \eta_{pq} = \eta_p \eta_q .$$

Let

$$\phi : T \times I \to G \quad \text{and} \quad \psi : T \times J \to G$$

be a mappings with

(ii) $$\phi(pq,i) = \phi(p,i\xi_q)\phi(q,i)$$

(iii) $$\psi(pq,j) = \psi(p,j)\psi(q,j\eta_p)$$

(iv) $$p_{ji\xi_p}\phi(p,i) = \psi(p,j)p_{j\eta_p i} .$$

Let us define a multiplication on $\Sigma = G \times I \times J \cup T$ with

(1) $$(a;i,j)(b;k,\ell) = (ap_{jk}b;i,\ell)$$

(2) $(a;i,j)p = (a\psi(p,j);i,j\eta_p)$

(3) $p(a;i,j) = (\phi(p,i)a;i\xi_p,j)$

(4) $pq = r \in T \Rightarrow pq = r \in \Sigma$

for all $i,k \in I$; $j,\ell \in J$; $p,q \in T$ and $a,b \in G$.

Then, Σ with a miltiplication defined above is a semigroup. We shall denote it by $M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$.

LEMMA 3.5. *A semigroup* S *has a kernel* K *which is a completely simple semigroup,* $S \setminus K$ *is a left simple semigroup and for each* $p \in S\setminus K$, $K \supseteq Kp$ *if and only if* S *is isomorphic with some* $M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$.

P r o o f. Let K be a completely simple kernel of S, $K = M(G;I,J;P)$, let $S\setminus K = T$ be a left simple semigroup and $K \subseteq Kp$ for each $p \in T$. Then by Theorem 1.1. |16| (see also Theorem 1.1. |13|) there exist functions ϕ,ψ,ξ and η with properties (i)-(iii) and a multiplication on S can be defined by (1)-(4). It remains to prove that the mapping η_p is a sirjection for each $p \in T$. For an arbitrary $j \in J$ we have that $(a;i,j) \in K$, i.e. there exists $(b;k,\ell) \in K$ such that

$$(a;i,j) = (b;k,\ell)p = (b\psi(p,\ell);k,\ell\eta_p) .$$

So $j = \ell\eta_p$ for some $\ell \in J$. Thus, η_p is a sirjection of J onto J.

Conversely, let η_p $(p \in T)$ be a mapping of J onto itself. Then for $(a;i,j) \in K$, $j = \ell\eta_p$ for some $\ell \in J$, so

$$(a;i,j) = (a[\psi(p,\ell)]^{-1};i,\ell)p \in Kp.$$

Thus, $K \subseteq Kp$ for each $p \in T$. The other conditions follow from the construction.

Let $K = I \times G$ be a left group, b a fixed element of G, $a \notin K$, $\xi : a \to \xi_a$, $\xi_a \in \mathcal{T}(I)$ and $\xi_a\xi_a = \text{const}$. Let us define a multiplication on $\Sigma = I \times G \cup \{a\}$ by:

$(i,x)(j,y) = (i,xy)$ $\qquad a(i,x) = (i\xi_a,bx)$

$(i,x)a = (i,xb)$ $\qquad aa = (i\xi_a\xi_a,b^2)$

for each $i,j \in I$; $x,y \in G$. Then, Σ with a miltiplication defined in this way is a semigroup, to be denoted by $M_2(G;I;a,b,\xi_a)$.

LEMMA 3.6. |2| . *A semigroup* S *contains a kernel* K *which is a left group,* $S \setminus K = \{a\}$ *and* $a^2 \in K$ *if and only if* S *is isomorphic with some* $M_2(G;I;a,b,\xi_a)$.

By Lemmas 3.4, 3.5 and 3.6. we have

THEOREM 3.1. *Every proper left ideal of a semigroup* S *is a completely simple semigroup if and only if one of the following conditions hold:*

1^o $S \cong M(G;I,J;P)$

2^o $S \cong M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$

3^o $S \cong M_2(G;I;a,b,\xi_a)$.

4. SEMIGROUPS IN WHICH SOME LEFT IDEAL IS A GROUP

THEOREM 4.1. *The following conditions on a semigroup* S *are equivalent:*

(i) *Some left ideal of* S *is a right group;*

(ii) *Some left ideal of* S *is a group;*

(iii) S *has a kernel which is a right group.*

P r o o f. (i) => (ii). Let a left ideal L of the semigroup S be a right group. Then L contains a left ideal G which is a group, so $SG = SGG \subset SLG \subset LG \subset G$. Thus G is a left ideal of S.

(ii) => (iii). See Corollary 2. |1|.

(iii) => (i) Follows immediately.

REMARK. Semigroups in which every proper left ideal is a group are described by Š.Schwarz, |17|. Semigroups in which some left ideal is a group are described by Ǵ. Čupona, |9|.

5. SEMIGROUPS IN WHICH EVERY PROPER LEFT IDEAL IS A RIGHT RIGHT GROUP

LEMMA 5.1. *Every proper left ideal of a semigroup* S *is a right group if and only if one the following conditions hold:*

1^o S *has a kernel* K *which is a right group and* $S \cap K = \emptyset$ *or* $S \setminus K = T$ *is a left simple semigroup and* $K \subseteq Ka$ *for each* $a \in T$ *;*

2^o S *has a kernel* K *which is a group and* $S \setminus K = \{a\}$, $a^2 \in K$.

P r o o f. If every proper left ideal of S is a right group, then by Lemma 3.4. there are three cases. Let the kernel K of S be a left group, $S \setminus K = \{a\}$, $a^2 \in K$. K has to be a right group also. Thus K is a group. The other cases and the converse follow immediately.

Let $K = G \times J$ be a right group and T a left simple semigroup such that $K \cap T = \emptyset$. Let $\eta : t \to \eta_t$ be a mapping of T into the semigroup of all epimorphisms of J onto itself and let $\phi : T \to G$ be a homomorphism. Let us define a multiplication on $\Sigma = G \times J \cup T$ by

$$(x,i)(y,j) = (xy,j) \qquad t(x,i) = (\phi(t)x,i)$$
$$(x,i)t = (x\phi(t),i\eta_t) \qquad ts = r \in T, \text{ then } ts = r \in \Sigma$$

for each $i,j \in J$; $x,y \in G$; $t,s \in T$. Then Σ , with a multiplication defined in this way, is a semigroup to be denoted by $M_3(G;J;T,\phi,\eta)$.

LEMMA 5.2. *A semigroup* S *contains a kernel* K *which is a right group,* $S \setminus K = T$ *is a left simple semigroup and* $K \subseteq Ka$ *for each* $a \in T$ *if and only if* S *is isomorphic with some* $M_3(G;J;T,\phi,\eta)$.

P r o o f. Similar to the proof of Lemma 3.4.

Let K be a group, b a fixed element of K and $a \notin K$. Let us define a multiplication * on $\Sigma = K \cup \{a\}$ by

$x*y = xy, \quad x,y \in K;$ $\qquad a*x = bx, \quad x \in K;$
$x*a = xb, \quad x \in K;$ $\qquad a*x = b^2$.

Then, Σ with this multiplication is a semigroup, to be denoted by $M_4(K;a,b)$.

It is easy to prove the following

LEMMA 5.3. *A semigroup* S *has a kernel* K *which is a group and* $S \setminus K = \{a\}$, $a^2 \in K$ *if and only if* S *is isomorphic with some* $M_4(K;a,b)$.

THEOREMA 5.1. *Every proper left ideal of a semigroup* S *is a right group if and only if one of the following condition hold :*

1° $S \cong M_3(G;J;T,\phi,\eta)$, *(T can be empty)*
2° $S \cong M_4(K;a,b)$.

P r o o f. Lemmas 5.1, 5.2. and 5.3. imply the assertion.

REMARK. Semigroups in which every proper left ideal is a left group are described by S.Bogdanović, |2|.

6. SEMIGROUPS IN WHICH EVERY PROPER BI-IDEAL IS A COMPLETELY SIMPLE SEMIGROUP

Let $K = M(G;I,J;P)$ be a Rees matrix semigroup and let T be a group such that $K \cap T = \emptyset$.

Let $\xi : p \to \xi_p$ be a mapping of T into the semigroup of all sirjection of I onto itself, and let $\eta : p \to \eta_p$ be a mapping of T into the semigroup of all sirjections of J onto itself, and let the conditions (i), from construction of Lemma 3.5., hold. Iet mappings $\phi : T \times I \to G$ and $\psi : T \times J \to G$ fulfil the conditions (ii)-(iv). We can define a multiplication on $\Sigma = K \cup T$ by (1)-(4). Then Σ with such a defined multiplication is a semigroup. Let us denote it by $M_5(G;I,J;P;T,\phi,\psi,\xi,\eta)$.

LEMMA 6.1. *A semigroup* S *has a kernel* K *which is a completely simple semigroup,* $S\setminus K$ *is a group and for each* $a\in S\setminus K$, $K\subset Ka\cap aK$ *if and only if* S *is isomorphic with some* $M_5(G;I,J;P;T,\phi,\psi,\xi,\eta)$.

THEOREM 6.1. *Every proper bi-ideal of a semigroup* S *is a completely simple semigroup if and only if one of the following conditions hold:*

1^o $S \cong M(G;I,J;P)$

2^o $S \cong M_5(G;I,J;P;T,\phi,\psi,\xi,\eta)$

3^o $S \cong M_4(K;a,b)$.

P r o o f. Let every proper bi-ideal of S be a completely simple semigroup. Then Lemma 3.4. S has a kernel K which is a completely simple semigroup and $S\setminus K$ is a left simple semigroup and $K\subseteq Ka$ for each $a\in S\setminus K$. By Lemma which is dual to the Lemma 3.4. $S\setminus K$ is, also, right simple and $K\subseteq aK$ for each $a\in S\setminus K$. Thus $S\setminus K$ is a group and $K\subseteq aK\cap Ka$ for each $a\in S\setminus K$. If K is a left group, $S\setminus K=\{a\}$, $a^2\in K$, then K has to be a right group and so K is a group and the assertion follows by Lemmas 6.1. and 5.3.

The converse follows immediately.

COROLLARY 6.1. *Every proper bi-ideal of* S *is a group if and only if one of the following conditions hold:*

1^o $S \cong M(G;I,J;P)$ *and* $|I|=2$, $|J|=1$ *or* $|I|=1$, $|J|=2$;

2^o $S \cong G\cup T$, $G\cap T=\emptyset$, G *and* T *are groups and the multiplication* $*$ *on* $G\cup T$ *is defined by using a homomorphism* $\phi: T\to G$ *in the following way*

$g*t = g\phi(t)$, $g\in G$, $t\in T$

$t*g = \phi(t)g$, $t\in T$, $g\in G$

$x*y = xy$ *in other cases,*

3^o $S \cong M_4(K;a,b)$

REMARK. By Theorem 1. |17| and Corollary 6.1. we have that every proper bi-ideal of S is a group if and only if S is an F-semigroup, |17| , i.e. semigroup in which every proper left and right subideal of S is a group.

7. SEMIGROUPS IN WHICH ALL PROPER SUBSEMIGROUPS ARE SIMPLE

The following theorem is given in |4|. Let M(2,r) denote a monogenic semigroup with the index 2 and period r.

THEOREM 7.1. *Every proper subsemigroup of a semigroup S is simple if and only if one of the following conditions hold:*

1° S *is* M(2,r) ;
2° $|S| = 2$;
3° S *is a completely simple periodic semigroup.*

P r o o f. Let every proper subsemigroup of S be simple, then S is a GE-semigroup (Theorem 2.3. |3|), also S is an M(2,r) or S is the union of periodic groups.

Let S be the union of periodic groups and $|S| > 2$.

If S is simple, then S is a completely simple periodic semigroup.

If S is not simple, then S has a completely simple kernel K, so (by Lemma 3.2) $S \setminus K = P$ is a left simple subsemigroup of S. Take idempotents, $e \in K$ and $f \in P$. They generated a semigroup T,

$$T = \langle e,f\rangle = \{e,f\} \cup \langle ef\rangle \cup \langle fe\rangle \cup \langle efe\rangle \cup \langle fef\rangle .$$

If $T \neq S$, then T is simple, thus

$$T = \langle ef\rangle \cup \langle fe\rangle \cup \langle efe\rangle \cup \langle fef\rangle \subset K.$$

i.e. $f \in K$. Contradiction. If T = S, then

$$T = \langle ef\rangle \cup \langle fe\rangle \cup \langle efe\rangle \cup \langle fef\rangle \subset K.$$

Contradiction.

Conversely, take case 3°, i.e. let S be a completely simple periodic semigroup and S' a subsemigroup of S. Then $S = M(G;I,J;P)$, G is a periodic group. Let $H_{ij} = \{(g;i,j): g \in G\}$. Changing the index, if necessary, we can get S' and H_{11} to have a nonempty interesection. $G' = \{g \in G: (g:1,1) \in S'\}$ is a subsemigroup of the periodic group G, thus, it is a subgroup of G. Immediately we can prove that $S \cap H_{ij} \neq \emptyset$ implies $S' \cap H_{i1} \neq \emptyset$.

In that case $S' \cap H_{i1} = \{(g;i,1): g \in G'\}$ and $S' \cap H_{ij} = \{(g;i,j): g \in G\}$ are provable. Also, if $S' \cap H_{ij} \neq \emptyset$ then the idempotent $(p_{ji}^{-1};i,j)$ is in S' and p_{ji} is in G'. Finally say $I' = \{i \in I: S' \cap H_{i1} \neq \emptyset\}$ and $J' = \{j \in J: S' \cap H_{ij} \neq \emptyset\}$ and therefore $S' = M(G';I',J';P')$.

Cases 1° and 2° follow immediately.

REMARK. Theorem 7.1. was given in |4|, but the proof was not completely correct. This is the reason wy it is given again with its complete proof. This theorem is a generalization of Proposition 1.1. |12| and the second part of the proof of the Theorem 7.1. is the same as the proof of Proposition 1.1. |12|.

COROLLARY 7.1. |15| . *Every proper subsemigroup of S is a group if and only if one of the following conditions hold:*

1° S *is* $M(2,r)$;

2° $|S| = 2$;

3° S *is a periodic group.*

8. SOME ISOMORPHISM THEOREMS

THEOREM 8.1. *Two semigroups* $S = M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$ *and* $S^* = M_1(G^*;I^*,J^*;P^*;T^*,\phi^*,\psi^*,\xi^*,\eta^*)$ *are isomorphic if and only if there is an isomorphism* $\omega: G \to G^*$, *a mapping* $i \mapsto u_i$ *of* I *into* G^*, *a mapping* $j \mapsto v_j$ *of* J *into* G^*, *bijections* $h: I \to I^*$ *and* $k: J \to J^*$ *and an isomorphism* $\Omega: T \to T^*$ *such that*

1) $p_{ji}\omega = v_j p^*_{jkih} u_i$

2) $\xi_p h = h\, \xi^*_{p\Omega}$

3) $\eta_p k = k \eta^*_{p\Omega}$

4) $\phi(p,i)\omega = u^{-1}_{i\xi_p} \phi^*(p\Omega, ih) u_i$

5) $\psi(p,j)\omega = v_j \psi^*(p\Omega, jk) v^{-1}_{j\eta_p}$

COROLLARY 8.1. *Two semigroups* $M_3(G;J;T,\phi,\eta)$ *and* $M_3(G^*;J^*;T^*,\phi^*,\eta^*)$ *are isomorphic if and only if there exists an isomorphism* $\omega : G \to G^*$, *a bijection* $k : J \to J^*$ *and an isomorphism* $\Omega : T \to T^*$ *such that* $\phi\omega = \Omega\phi^*$ *and* $\eta_p k = k\eta^*_{p\Omega}$.

THEOREM 8.2. *Two semigroups* $M_2(G;I;a,b,\xi_a)$ *and* $M_2(G^*;I^*;a^*,b^*,\xi_{a^*})$ *are isomorphic if and only if there is an isomorphism* $\omega : G \to G^*$ *and a bijection* $h : I \to I^*$ *such that* $b\omega = b^*$ *and* $\xi_a h = h\xi^*_{a^*}$.

COROLLARY 8.2. *Two semigroups* $M_4(G;a,b)$ *and* $M_4(G^*;a^*,b^*)$ *are isomorphic if and only if there is an isomorphism* $\omega : G \to G^*$ *such that* $\omega(b) = b^*$.

We have given some isomorphism theorems but we shall only prove Theorem 8.1. because the other proofs are similar to this one.

P r o o f of Theorem 8.1. Let $f : S \to S^*$ be an isomorphism $G \times I \times J$ is a kernel in S and $G^* \times I^* \times J^*$ is a kernel in S^*, therefore $G \times I \times J \cong G^* \times I^* \times J^*$. Now, by Theorem 2.8. |10| , there exists an isomorphism $\omega : G \to G^*$, bijections $h : I \to I^*$, $k : J \to J^*$ and elements $u_i, v_j \in G$ $(i \in I,\ j \in J)$ such that condition 1) is satisfied. For $p \in T$ and $(a;i,j) \in G \times I \times J$ we have

$$(p(a;i,j))f = (\phi(p,i)a; i\xi_p, j)f =$$

$$= [u_{i\xi_p} (\phi(p,i)a)\omega v_j; i\xi_p h, jk] \in G^* \times I^* \times J^*$$

and

$$(pf)((a;i,j)f) = p\Omega[u_i(a\omega)v_j; ih, jk)] =$$

$$= [\phi^*(p\Omega, ih)u_i(a\omega)v_j; ih\xi^*_{p\Omega}, jk]$$

so $\xi_p h = h\xi^*_{p\Omega}$, i.e. condition 2) holds. The proof of 3) is similar. Further,

$$u_{i\xi_p}(\phi(p,i)a)\omega v_j = \phi^*(p\Omega,ih)u_i(a\omega)v_j$$

therefore

$$\phi(p,i)\omega = u_{i\xi_p}^{-1}\phi^*(p\Omega,ih)u_i \ .$$

Hence, condition 4) is satisfied. The proof of 5) is similar.

Conversely, let us define a mapping $f : S \to S^*$ by

$$xf = \begin{cases} [u_i(a\omega)v_j; ih, jk] , & x \in G \times I \times J \\ x\Omega , & x \in T . \end{cases}$$

f is an isomorphism. For $x,y \in G \times I \times J$ f is an isomorphism by Theorem 2.8. |10|. Ω is an isomorphism of T onto T* . It remains to prove that f is an isomorphism for $p \in T$ and $(a;i,j) \in G \times I \times J$. Indeed,

$$(p(a;i,j))f = (\phi(p,i)a; i\xi_p, j)f =$$

$$= [u_{i\xi_p}(\phi(p,i)a)\omega v_j; i\xi_p h , jk] =$$

$$= [u_{i\xi_p}\phi(p,i)\omega(a\omega)v_j; ih\xi^*_{p\Omega}, jk] =$$

$$= [u_{i\xi_p}u^{-1}_{i\xi_p}\phi^*(p\Omega,ih)u_i(a\omega)v_j; ih\xi^*_p, jk] =$$

$$= [\phi^*(p\Omega,ih)u_i(a\omega)v_j; ih\xi^*_{p\Omega}, jk]$$

and

$$(pf)((a;i,j)f) = p\Omega[u_i(a\omega)v_j; ih, jk] =$$

$$= [\phi^*(p\Omega,ih)u_i(a\omega)v_j; ih\xi^*_{p\Omega}, jk] \ :$$

Thus $(p(a;i,j))f = (pf)((a;i,j)f)$. The proof of $((a;i,j)p)f = ((a;i,j)f)(pf)$ is similar. Thus, f is as isomorphism.

REMARK. If T and T* are partial semigroups, then a similar Theorem to Theorem 8.1. can be given. In that case instead of the isomorphism $\Omega : T \to T^*$ we have a partial isomorphism. Such a theorem (without restriction for η_p) is given without a proof in |16|.

REFERENCES

|1| Bogdanović,S., *Semigroups in which some bi-ideal is a group, Zbornik radova PMF Novi Sad, Vol. 11(1981), 261-266.*

|2| Bogdanović,S., *Semigroups in which every proper left ideal is a left group, Notes on Semigroups VIII, K.Marx University, Budapest.*

|3| Bogdanović,S., *Sur les demi-groupes dans lesquels tous les sousdemi-groupes propres sont idempotents, Math. Seminar Notes, 9(1981), 17-24, Kobe Univ.*

|4| Bogdanović, S., *Sur les demi-groupes dans lesquels tous les sousdemi-groupes sont idempotents II, Mat. Vesnik 5(18)(33),1981, 239-243.*

|5| Clifford,A.H. and G.B.Preston *The algebraic theory of semigroups, Amer. Math. Soc. 1961.*

|6| Clifford,A.H., *Semigroups containing minimal ideals, Amer. J. Math. 70(1948), 521-526.*

|7| Clifford,A.H., *Extensions of semigroups, Trans.Amer.Math.Soc. 68(1950), 165-173.*

|8| Čupona,Ǵ., *On semigroups S in which each proper subset Sx is a group, Glasnik Mat. Fiz. - Astr. 18(1963).*

|9| Čupona,Ǵ., *Semigroups in which some left ideal is a groups, God. Zbornik PMF-a, Skopje, T.14 (1963), 15-17.*

|10| Howie,J., *An Introduction to Semigroup Theory, Acad.Press 1976.*

|11| Hrmová,R., *Polugrupy v ktorých každý lývý vlastný ideál je groupu, Mat. Fiz. Časop. 1961, 11, No 1, 75-80.*

|12| Lallement,G., *On ideal extension of completely simple semigroups, Semigroup Forum Vol. 24(1982), 223-230.*

|13| Milić,S. and V.Pavlović, *Semigroups in which some ideal is a completely simple semigroup, Publ.Inst.Math. 30(44), 1982, 123-130.*

|14| Petrich,M., *Introduction to Semigroups, Merill Publ.Comp. 1973.*

|15| Pollák,G. and L.Rédei, *Die Halbgruppen, deren alle echten Teilhalbgruppen Gruppen Sind, Publ. Math. Debrecen 6(1959), 125-131.*

|16| Protić,P. and S.Bogdanović, *On a class of semigroups, Algebraic Conference 1981, Novi Sad, 113-119.*

|17| Schwarz,Š., *Semigroups in which every proper subideal is a group, Acta Sci. Math. Szeged, 21(1960), 125-131.*

|18| Steinfeld,O., *Quasi-ideals in rings and semigroups, Akadémiai Kiadó, 1978.*

Received by the editors January 20,1983.

REZIME

POLUGRUPE SA POTPUNO PROSTIM JEZGROM

U ovom radu daju se neke nove karakterizacije pomoću bi-ideala i levih ideala za polugrupe sa potpuno prostim jezgrom. Takodje, u ovom radu razmatraju se polugrupe u kojima svaki pravi levi ideal (bi-ideal) jeste potpuno prosta polugrupa i teoreme o izomorfizmu ovakvih polugrupa su date.