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FUZZY CONGRUENCE RELATIONS AND CONSTRUCTIONS OF ALGEBRAS

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ABSTRACT

The Boolean extension of an algebra (|1|) is here given as a collection of fuzzy sets. It is shown that fuzzy relations, especially fuzzy congruence relations (|2|,|3|) are closely related to those structures. Some applications of fuzzy relations in constructing new algebras and extensions are given.

1. A fuzzy binary relation ρ on the set A, is a fuzzy subset of $A \times A$, with the membership function $\mu_\rho : A \times A \to B$, where $\mathcal{B} = \langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra.

2. A fuzzy equivalence relation ρ on A is a fuzzy binary relation on A, which satisfies:

a) $(\forall a \in A)(\mu_\rho(a,a) = 1)$ (ρ is reflexive);
b) $(\forall a,b \in A)(\mu_\rho(a,b) = \mu_\rho(b,a))$ (ρ is symmetric);
c) $\mu_\rho(a,b) \geq \bigvee_{c \in A} (\mu_\rho(a,c) \wedge \mu_\rho(c,b))$, for all $a,b \in A$. (ρ is transitive).

3. If $\mathcal{A} = \langle A, \mathcal{O}\rangle$ is an algebra then a fuzzy binary relation θ on A is a fuzzy congruence relation on $\mathcal{A}$, iff it is a fuzzy equivalence relation on A, satisfying the substitution property:

If $\mu_\theta(a_i,b_i) = p_i$, $i = 1,\ldots,n$, $(p_i \in B)$, then for every n-ary operation $f \in \mathcal{O}$,

$$(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \geq \bigwedge_{i=1}^{n} p_i \quad (|2|).$$

DEFINITION 1. |1| *Let* $\mathcal{A} = \langle A, O\rangle$ *be an algebra, and* $\mathcal{B}$ *the complete Boolean algebra. The collection* $A(B)$ *of fuzzy sets* X *on* A *is said to be a Boolean extension of* $\mathcal{A}$ *iff for every* $X \in A(B)$ *:*

1. $a \neq b$, $a, b \in A$, *implies:* $\mu_X(a) \wedge \mu_X(b) = 0$, *and*

2. $\bigvee_{a\in A} \mu_X(a) = 1.$

$\mathcal{A}(\mathcal{B}) = \langle A(B), O\rangle$ is a new algebra, with the operations defined in the following way:

$f(X_1, X_2, \ldots, X_n) = Y$ where, for $a \in A$,

$\mu_Y(a) = \bigvee_{f(a_1,\ldots,a_n)=a} (\mu_{X_1}(a_1) \wedge \ldots \wedge \mu_{X_n}(a_n))$, and the supremum runs over all n-tuples $(a_1, \ldots, a_n)$, such that $f(a_1, \ldots, a_n) = a$.

It is easy to check that the operations are well defined, i.e. that Y is also a fuzzy set satisfying 1) and 2). The supremum used in defining the operations explains why $\mathcal{B}$ has to be complete (provided that A is not finite).

REMARK

If $\mathcal{B} \simeq \mathcal{P}(I)$ (as the Boolean algebra), then $\mathcal{A}(\mathcal{B})$ is isomorphic to the direct power $\mathcal{A}^I$ (|4|).

In the following we shall consider some special fuzzy relations on $\mathcal{A}(\mathcal{B})$.

PROPOSITION 1. *Let* $\mathcal{A}$ *be an algebra, and* $\mathcal{B}$ *the complete Boolean algebra. Let* θ *be an (ordinary) congruence relation* $\mathcal{A}$. *Then the fuzzy relation* $\underline{\theta}$, *given by*

$$\mu_{\underline{\theta}}(X,Y) = \bigvee_{(a,b)\in\theta} (\mu_X(a) \wedge \mu_Y(b)), \quad X, Y \in A(B),$$

is a fuzzy congruence relation on $\mathcal{A}(\mathcal{B})$.

P r o o f. $\underline{\theta}$ is a fuzzy equivalence relation on A(B): It is obvious that $\underline{\theta}$ is reflexive and symmetric in the sense of a) and b) in 2. We shall here prove that it is transitive:

$$\mu_{\underline{\theta}}(X,Z) \wedge \mu_{\underline{\theta}}(Z,Y) =$$

$$= \bigvee_{(a,c_1)\in\theta} (\mu_X(a) \wedge \mu_Z(c_1)) \wedge \bigvee_{(c_2,b)\in\theta} (\mu_Z(c_2) \wedge \mu_Y(b)) =$$

$$= \bigvee (\mu_X(a) \wedge \mu_Z(c_1) \wedge \mu_Z(c_2) \wedge \mu_Y(b) ; (a,c_1) \in \theta , (c_2,b) \in \theta) =$$

$$= \bigvee (\mu_X(a) \wedge \mu_Z(c) \wedge \mu_Y(b); \ (a,c) \in \theta, \ (c,b) \in \theta) \leq$$

$$\bigvee_{(a,b)\in\theta} (\mu_X(a) \wedge \mu_Y(b)) = \mu_{\underline{\theta}}(X,Y),$$

by 1), Definition 1., and since θ is transitive. Now let $\mu_{\underline{\theta}}(X_i,Y_i) = r_i$, $i=1,\dots,n$, and for an n-ary operation f, $\mu_{\underline{\theta}}(f(X_1,\dots,X_n),f(Y_1,\dots,Y_n)) = r$. Then,

$$\bigwedge_{i=1}^{n} r_i = \bigwedge_{i=1}^{n} (\bigvee_{(a,b)\in\theta} (\mu_{X_i}(a) \wedge \mu_{Y_i}(b)) =$$

$$\bigvee (\bigwedge_{i=1}^{n} (\mu_{X_i}(a_i) \wedge \mu_{Y_i}(b_i)); (a_i,b_i) \in \theta, \ i=1,\dots,n) \leq$$

$$\bigvee (\bigwedge_{i=1}^{n} (\mu_{X_i}(a_i) \wedge \mu_{Y_i}(b_i)); (f(a_1,\dots,a_n),f(b_1,\dots,b_n)) \in \theta) =$$

$$\bigvee_{(a,b)\in\theta} (\bigvee (\bigwedge_{i=1}^{n} \mu_{X_i}(a_i); f(a_1,\dots,a_n)=a) \wedge$$

$$\bigvee (\bigwedge_{i=1}^{n} \mu_{Y_i}(b_i); f(b_1,\dots,b_n)=b) = \mu_{\underline{\theta}}(f(X_1,\dots,X_n),f(Y_1,\dots,Y_n)),$$

which proves the substitution property.

If θ is the equality on A, then the corresponding fuzzy congruence relation on A(B) can be given as follows:

COROLLARY 2. *Fuzzy relation* $\underline{\sigma}$ *on* A(B), *given by its membership function*

$$\mu_{\underline{\sigma}}(X,Y) = \bigvee_{a\in A} (\mu_X(a) \wedge \mu_Y(a)),$$

is a fuzzy congruence relation on A(B).

* * *

In the following propositions we shall describe the use of fuzzy congruence relations in some constructions of algebras, based on Boolean extensions.

DEFINITION 2. *Let* A(B) *be the Boolean extension of* A, $\underline{\sigma}$ *the fuzzy congruence relation on* A(B) *(defined in Corollary 2.)*

and F *any filter in* B. *The reduced extension* $A(B)_F$ *of* A *is an algebra in which*

$$A(B)_F = \{\sigma(X_F)\ ;\quad X \in A(B)\}\quad ,\ \textit{and}$$

$$\sigma(X)_F = \{Y; Y \in A(B)\quad \textit{and}\quad \mu_{\underline{\sigma}}(X,Y) \in F\}.$$

The operations on these collections of fuzzy sets are defined in the natural way:

$$f(\sigma(X_1)_F,\dots,\sigma(X_n)_F) \stackrel{\text{def}}{=\!=} \sigma(f(X_1,\dots,X_n))_F\ ,$$

and it is easy to show that the resulting element does not depend on the representatives, the fuzzy sets $X_1,\dots,X_n$.

Clearly, the following lemma holds.

LEMMA 3. *If* F *is a principal ultrafilter in* B, *then*

$$A(B)_F \cong A\ .$$

P r o o f. The isomorphism is given by the mapping $h : A \to A(B)_F$, where $h(a) = \sigma(X)_F$ and $\mu_X(a) = 1$.

Let p and q be two arbitrary elements of a Boolean algebra B. Denote the Boolean expression $(p \vee q') \wedge (p' \vee q)$ by $p \odot q$.

PROPOSITION 4. *Let* $A = \langle\{a,b\}, 0\rangle$ *be a two-element algebra, and* B *an arbitrary Boolean algebra. Then there is a* 1 - 1 *map of* B *into* $A(B)(p \to X_p)$ *such that if* $\underline{\sigma}$ *is a fuzzy congruence relation defined in Corollary 2, then for all* $p,q \in B$, $\mu_{\underline{\sigma}}(X_p,X_q) = p \odot q$.

P r o o f. If $X \in A(B)$, then $\mu_X(a) = p$ implies $\mu_X(b) = p'$. Hence, every $p \in B$ uniquely determines one X, and if $p \neq q$, then clearly $X_p \neq X_q$. Now, $\mu_{\underline{\sigma}}(X_p,X_q) = (\mu_{X_p}(a) \wedge \mu_{X_q}(a)) \vee \mu(X_p(b) \wedge \mu_{X_q}(b)) = (p \wedge q) \vee (p' \wedge q') = (p \wedge q') \wedge (p' \wedge q) = p \odot q$.

Now let $p^k = \begin{cases} p', & \text{if } k = 0 \\ p, & \text{if } k = 1 \end{cases}$, $p \in B$, $k \in \{0,1\}$.

A converse of the preceeding proposition can be formulated as follows.

PROPOSITION 5. *Let* $A = \langle A, \sigma\rangle$ *be an algebra,* B *any Boolean algebra, and suppose that there is and* 1—1 *map* x_p *of* B *onto* A *, and a fuzzy congruence relation* $\underline{\sigma}$ *on* A *, with* $\mu_{\sigma}(x_p, x_q) = p \Theta q$ *. Also let* $f(x_{p_1}, \ldots, x_{p_n}) = x_q$, $(f \in O)$, *where*

$$\text{(a)} \qquad q = \bigvee (p_1^{i_1} \wedge \ldots \wedge p_n^{i_n} \; ; \; f(X_{F^{i_1}}, \ldots, X_{F^{i_n}}) = X_F)$$

$$A/_{\sigma_F} = \langle \{X_F, X_{F'}\}, O\rangle$$

and X_F *is a class to which* X_1 $(1 \in B)$ *belongs.*

Then there is a two element algebra A_1 *, such that*

$$A_1(B) \cong A \; .$$

P r o o f. Let $A_1 = A/_{\sigma_F}$ and denote the elements of $A_1(B)$ by X_p, if $\mu_{X_p}(X_F) = p$. The map h of $A_1(B)$ onto A: $h(X_p) = x_p$, is an isomorphism. Really, it is 1 - 1 by definition. Also,

$$h\,(f(X_{p_1}, \ldots, X_{p_n})) = h(X_q), \quad \text{where,}$$

$$\mu_{X_q}(X_F) = \bigvee \left(\mu_{X_{p_1}}(X_{F^{i_1}}) \wedge \ldots \wedge \mu_{X_{p_n}}(X_{F^{i_n}}) \; ;\right.$$

$$\left. f(X_{F^{i_1}}, \ldots, X_{F^{i_n}}) = X_F\right) = \bigvee (p_1^{i_1} \wedge \ldots \wedge p_n^{i_n} \; ;$$

$$f(X_{F^{i_1}}, \ldots, X_{F^{i_n}}) = X_F, \quad \text{and thus,}$$

by (a), $h(X_q) = x_q = f(x_{p_1}, \ldots, x_{p_n}) = f(h(X_{p_1}), \ldots, h(X_{p_n}))$.

We shall describe now some fuzzy properties of induced notions, and then we shall use them in describing one construction of algebras. First, we repeat some definitions from |3|.

i) A fuzzy function from A to A_1 is a fuzzy relation $\underline{f}$ from A to A_1 such that

- for every $a \in A$, there is exactly one $X \in A_1$, such that $\mu_{\underline{f}}(a, X) = 1$;

1) σ_F is a regular binary relation: $(a,b) \in \sigma_F$ iff $\mu_{\underline{\sigma}}(a,b) \in F$ (see |4|).

- every $p \in B$, $p \neq 0$, appears once at most as a value of $\mu_{\underline{f}}(a,X)$, $X \in A_1$.

ii) A fuzzy function $\underline{h}$ from the algebra $\mathcal{A}_1 = \langle A_1, \mathcal{O} \rangle$ to the algebra $\mathcal{A}_2 = \langle A_2, \mathcal{O} \rangle$ is said to be a fuzzy homomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$, if from

$\mu_{\underline{h}}(a_i, b_i) = p_i$, $a_i \in A_1, b_i \in A_2$, $i=1,\ldots,n$, it follows that

$$\mu_{\underline{h}}(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \in ([\bigwedge_{i=1}^{n} p_i)^*, \quad f \in \mathcal{O} .$$

LEMMA 6. *Let $X \in A(B)$, $p \in B$, and $\underline{\sigma}$ the fuzzy congruence relation defined in Corollary 2. Then there is $\underline{Y} \in A(B)$ such that $\mu_{\underline{\sigma}}(\underline{X}, \underline{Y}) = p$.*

P r o o f. Let a and b be two arbitrary elements from algebra A, and let $\mu_{\underline{X}}(a) = p_a$, $\mu_{\underline{X}}(b) = p_b$. Define $\underline{Y} \in A(B)$ as follows:

$$\mu_{\underline{Y}}(a) = (p \wedge p_a) \vee \bigvee_{x \neq a} (p' \wedge p_x), \ (x \in A), \ \mu_{\underline{Y}}(b) =$$

$$= (p \wedge p_b) \vee (p' \wedge p_a) \ , \ \text{and}$$

$$\mu_{\underline{Y}}(x) = p \wedge p_x, \ x \neq a, \ x \neq b.$$

Now it is easy to show that $\underline{Y} \in A(B)$, and that

$$\mu_{\underline{\sigma}}(\underline{X}, \underline{Y}) = p.$$

Now, starting with $\mathcal{A}(B)$, we can construct a new algebra $\mathcal{A}$, by means of the fuzzy congruence relation $\underline{\sigma}$ (generalizing the notion of a quotient algebra). Let

$$A_{\underline{\sigma}} = \bigcup_{p \in B} A(B)_{[p)} \ .$$

The operations are defined as in |3|:

*) $[p)$ is a principal filter in B, generated by p.

$$f(\sigma(a_1)_{[p)},\ldots,\sigma(a_n)_{[p)}) = \sigma(a)_{[\bigwedge_{i=1}^{n} p_i)},$$

$a = f(a_1,\ldots,a_n)$, $f \in \mathcal{O}$.

The fact that p runs over all elements of B, is the consequence of Lemma 6.

PROPOSITION 7. *Algebra* $A_{\underline{\sigma}}$ *is a homomorphic image of* $A(B)$.

P r o o f. The fuzzy relation $\underline{h}$ from $A(B)$ to $A_{\underline{\sigma}}$, defined as

$$\mu_{\underline{h}}(\underline{x},X) = \begin{cases} V(p_i,\ \sigma(\underline{x})_{[p_i)} = X)\ , & \text{if } \underline{x} \in X \\ 0\ , & \text{otherwise,} \end{cases}$$

is the required fuzzy homomorphism, as in |3|, and by Lemma 6.

If $A(B)$ is the extension of a two-element algebra A, then by Lemma 6, and by Proposition 4, every $p \in B$ appears exactly ones as a value $\mu_{\underline{\sigma}}(a,\underline{x})$, for an $\underline{a} \in A(B)$. Consider now the fuzzy relation $\underline{g}$ from A to $A(B)$:

$$\mu_{\underline{g}}(a,\underline{X}) = \mu_{\underline{\sigma}}(\underline{P},\underline{X})\ ,\ \text{where } \underline{P} \in A(B),\ \text{and } \mu_{\underline{P}}(a) = 1.$$

PROPOSITION 8. *A fuzzy relation* $\underline{g}$ *is a fuzzy homomorphism from a two-element algebra* A *to* $A(B)$.

P r o o f. By construction $\underline{g}$ is a fuzzy function. It satisfies the supstitution property:

Clearly, $\mu_{\underline{g}}(a,\underline{X}) = \mu_{\underline{X}}(a)$. Thus,

$\mu_{\underline{\sigma}}(a_i,\underline{X}_i) = \mu_{\underline{X}_i}(a_i)$, $i=1,\ldots,n$, and

$\mu_{\underline{\sigma}}(f(a_1,\ldots,a_n),f(\underline{X}_1,\ldots,\underline{X}_n)) = \mu_{\underline{Z}}(f(a_1,\ldots,a_n))$,

$\underline{Z} = f(\underline{X}_1,\ldots,\underline{X}_n)$. Now, since $\underline{Z}$ is from A(B),

$$\mu_{\underline{Z}}(f(a_1,\ldots,a_n)) = \vee(\mu_{\underline{X}_1}(b_1)\wedge\ldots\wedge\mu_{\underline{X}_n}(b_n)),\ \text{for all}$$

$b_1,\ldots,b_n$ such that $f(a_1,\ldots,a_n) = f(b_1,\ldots,b_n)$, and this obviously is not less than $\mu_{\underline{X}_1}(a_1)\wedge\ldots\wedge\mu_{\underline{X}_n}(a_n)$.

We finally give one example of the fuzzy mappings from the Propositions 7. and 8., using an arbitrary Boolean algebra B:

$A = \langle\{a,b\},*\rangle \qquad B = \{0,1,p,p',q,q',\ldots\}$

*	a	b
a	a	b
b	b	a

$A(B) = \{\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \underline{X}_1,\ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \underline{X}_o,\ \begin{pmatrix} a & b \\ p & p' \end{pmatrix} = \underline{X}_p,\ \ldots\}$. $\underline{X}_p * \underline{X}_q = \begin{pmatrix} a & b \\ p\odot q & (p\odot q)' \end{pmatrix}$.

$\underline{\sigma}$	$\underline{X}_o$	$\underline{X}_1$	$\underline{X}_p$	$\underline{X}_{p'}$	$\underline{X}_q$	...
$\underline{X}_o$	1	0	p'	p	q'	...
$\underline{X}_1$	0	1	p	p'	q	...
$\underline{X}_p$	p'	p	1	0	$p\odot q$	...
$\underline{X}_{p'}$	p	p'	0	1	$p'\odot q$	...
$\underline{X}_q$	q'	q	$p\odot q$	$p'\odot q$	1	...
.	.	.	.	.	.	.

$\underline{g}$	$\underline{X}_o$	$\underline{X}_1$	$\underline{X}_p$	$\underline{X}_{p'}$	$\underline{X}_q$	...
a	0	1	p	p'	q	...
b	1	0	p'	p	q'	...

$\underline{h}$	$\{\underline{X}_o\}$	$\{\underline{X}_1\}$	$\{\underline{X}_p\}$		$\{\underline{X}_o,\underline{X}_{p'},\ldots\}$	$\{\underline{X}_1,\underline{X}_p,\ldots\}$	...	$\{\underline{X}_o,\underline{X}_p,\ldots\}$	$\{\underline{X}_1,\underline{X}_{p'}\}$	
$\underline{X}_o$	1	0	0	...	p	0	...	p'	0	...
$\underline{X}_1$	0	1	0	...	0	p	...	0	p'	...
$\underline{X}_p$	0	0	1	...	0	p	...	p'	0	...
$\underline{X}_{p'}$	0	0	0	...	p	0	...	0	p'	...

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REZIME

RASPLINUTE KONGRUENCIJE I KONSTRUKCIJE ALGEBRI

U radu se pokazuje da se koncept rasplinutih skupova može proširiti i na Bulove ekstenzije algebri i da se na njima mogu posmatrati rasplinute relacije. Opisana je jedna klasa rasplinutih relacija, kongruencija na tim algebrama i pokazano je da se one mogu koristiti za konstruisanje i opisivanje nekih konstrukcija algebri.