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SOME APPLICATIONS OF A FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT

In [3] a fixed point theorem for multivalued mappings in not necessarily locally convex topological vector spaces is proved. Here we obtain, by using this fixed point theorem and the dual result, two theorems on the coincidence point, a result on the equilibrium state in a special noncooperative game and three existence theorems for some classes of equations.

1. NOTATIONS AND DEFINITIONS

Recently some fixed point theorems for multivalued mappings in not necessarily locally convex topological vector spaces have been proved ([3], [4], [5], [8], [9], [10], [11], [13]). Some applications in the theory of optimization are given in [6] and [12]. This paper contains some further applications of fixed point theorems from [3] and [11]. The following notations and definitions are taken from [15] and [6]. In this paper it will be assumed that all topological vector spaces are Hausdorff. If X is a topological space, by 2^X we shall denote the family of all nonempty subsets of X and by 2^X_c the family of

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all nonempty closed subsets of X. The family of all nonempty, closed and convex subsets of X will be denoted by $\mathcal{K}(X)$. If X is a topological vector space and $A\subseteq X$ then co A denotes the convex hull of A.

Let $f:X\to Y$, where X and Y are topological spaces. Then for every $A\subseteq X$, $B\subseteq Y$:

$$f(A)=\bigcup_{x\in A} f(x), \quad f^{-1}(B)=\{x|x\in X,\ f(x)\cap B\neq\emptyset\} .$$

The mapping $f:X\to Y$ is upper semicontinuous if and only if for each closed set $B\subseteq Y$, the set $f^{-1}(B)$ is a closed subset of X.

DEFINITION. *Let* E *be a topological vector space,* $\mathcal{U}$ *be the fundamental family of neighbourhoods of zero in* E *and* $K\subseteq E$. *We say that the set* K *is of Zima's type if and only if for every* $V\in\mathcal{U}$ *there exists* $U\in\mathcal{U}$ *so that* $co(U\cap(K-K))\subseteq V$.

Some examples of subsets of Zima's type in not necessarily locally convex topological vector spaces are given in [6]. Let E be a vector space and $\|\ \|^*:E\to[0,\infty)$ so that the following conditions are satisfied:

1. $\|x\|^*=0 \iff x=0$.
2. $\|x\|^*=\|-x\|^*$, for every $x\in E$.
3. $\|x+y\|^*\leq\|x\|^*+\|y\|^*$, for every $x,y\in E$.
4. If $\|x_n-x_o\|^*\to 0$ and $\lambda_n\to\lambda_o$, when $n\to\infty$ then $\|\lambda_n x_n - \lambda_o x_o\|^*\to 0$.

Then $(E,\|\ \|^*)$ is a paranormed space. This is a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family $\mathcal{V}=\{V_r\}_{r>0}$, where:

$$V_r=\{x|x\in E,\ \|x\|^*<r\} .$$

In [16] Zima proved a generalization of the Schauder fixed point theorem for the mapping $f:K\to K$, where K is a subset of E and $(E, \| \ \|^*)$ is a paranormed space and there exists $C > 0$ so that

(1) $\quad \| tx \|^* \leq C\, t \| x \|^*$, for every $t \in [0,1]$ and $x \in f(K)-f(K)$.

It is easy to see that the inequality (1) implies that f(K) is of Zima's type. In [6] an example of E and K is given, where $K \subseteq E$ and $(E, \| \ \|^*)$ is a non-locally convex paranormed space so that :

$$\| tx \|^* \leq C\, t \| x \|^* \text{ for every } t \in [0,1] \text{ and } x \in K-K .$$

An example of K and E is given in [9] and [16].

In [3] the following fixed point theorem is proved.

THEOREM A. *Let* E *be a topological space,* $\mathcal{U}$ *the fundamental system of neighbourhoods of zero in* E, K *a closed and convex subset of* E, $f: K \to \mathcal{R}(K)$ *an upper semicontinuous mapping such that* $\overline{f(K)}$ *is compact and* f(K) *is of Zima's type. Then there exists* $x \in K$ *so that* $x \in f(x)$.

2. TWO THEOREMS ON THE COINCIDENCE POINT

Using the same method as in [15], we shall prove the following coincidence theorem.

THEOREM 1. *Let* E *be a topological vector space,* S *a nonempty closed and convex subset of* E, K *a compact subset of* S, H *a topological space and* $\phi: S \to 2^H_C$, $\psi: K \to 2^H_C$ *upper semicontinuous mappings such that* $\psi^{-1}(\phi(S))$ *is of Zima's type. Let for every* $x \in S$:

(i) $\quad \phi(x) \cap \psi(K) \neq \emptyset$.

(ii) $\quad \psi^{-1}(\phi(x))$ *be convex.*

If H *is regular or* H *is Hausdorff and* $\psi(x)$ *is compact for every* $x \in K$, *there exists* $x_0 \in K$ *such that* $\phi(x_0) \cap \psi(x_0) \neq \emptyset$.

P r o o f. Let us define, as in [15], the mapping $\hat{\phi}:S\to 2^K_c$ in the following way:

$$\hat{\phi}(x) = \psi^{-1}(\phi(x)), \quad x\in S .$$

It remains to be proved that $\hat{\phi}$ satisfies all the conditions of Theorem A. Since $\psi^{-1}(\phi(S))$ is of Zima's type, we shall prove that $\hat{\phi}$ is upper semicontinuous and $\hat{\phi}(x)\in R(K)$ for every $x\in S$. Since (ii) holds, the relation $\hat{\phi}(x)\in R(K)$ follows from the upper semicontinuity of ψ and the closedness of the set $\phi(x)$. The upper semicontinuity of $\hat{\phi}$ follows as in [15], since

$$\begin{aligned}\hat{\phi}^{-1}(A) &= \{x|x\in S,\ \hat{\phi}(x)\cap A\neq\emptyset\} = \\ &= \{x|x\in S,\ \psi^{-1}(\phi(x))\cap A\neq\emptyset\} = \\ &= \{x|x\in S,\ \phi(x)\cap\psi(A)\neq\emptyset\}\end{aligned}$$

where A is a closed subset of K.

COROLLARY 1. *Let X be a topological vector space, L a nonempty, closed and convex subset of X, $f:L\to R(X)$ an upper semicontinuous mapping such that $\overline{f(L)}$ is compact, and G a linear one to one mapping from X onto X such that G and G^{-1} are continuous and $f(L)\subseteq G(L)$. If f(L) is of Zima's type, there exists $x\in L$ such that $G(x)\in f(x)$.*

P r o o f. Let $H=X, S=L$, $K=G^{-1}\overline{f(L)}$, $\phi=f$ and $\psi=G$. From the compactness of $\overline{f(L)}$, it follows that K is a compact subset of L. Since G^{-1} is a linear mapping and $f(x)\in R(X)$ for every $x\in L$, it follows that $G^{-1}f(x)\in R(X)$.

From $f(L)\subseteq G(L)$ it follows that (i) is satisfied. Let us prove that $G^{-1}(f(S))$ is of Zima's type. By $\mathcal{U}$ we shall denote the family of all neighbourhoods of zero in X and let $V\in\mathcal{U}$. We shall prove that there exists $U\in\mathcal{U}$ so that:

$$co(U\cap(G^{-1}f(S) - G^{-1}f(S)))\subseteq V .$$

Since G^{-1} is continuous and linear, there exists $V'\in\mathcal{U}$ so that $G^{-1}(V')\subseteq V$. Further, the set f(L) is of Zima's type and so there exists $U' \in \mathcal{U}$ such that:

$$co(U' \cap (f(S)-f(S))) \subseteq V' .$$

From the linearity of the mapping G^{-1}, we have that:

$$G^{-1}(co(U' \cap (f(S)-f(S)))) = co(G^{-1}(U' \cap (f(S)-f(S)))) .$$

This implies that:

$$(2) \quad co(G^{-1}(U') \cap G^{-1}(f(S)-f(S))) \subseteq G^{-1}(V') \subseteq V .$$

Further, the mapping G is continuous and so there exists $U \in \mathcal{U}$ such that $GU \subseteq U'$. Hence $U \subseteq G^{-1}(U')$, and from (2) we obtain that:

$$co(U \cap (G^{-1}f(S)-G^{-1}f(S))) \subseteq co(G^{-1}(U') \cap (G^{-1}f(S)-G^{-1}f(S))) \subseteq V.$$

Using the method of duality and Theorem A, in [11] the following fixed point theorem is proved:

THEOREM B. *Let* L *be a nonempty compact subset of Zima's type of a topological vector space* E, $f:L \to 2^E$ *an upper semicontinuous mapping such that* $L \subseteq f(L)$, $f(x) = \overline{f(x)}$ *for every* $x \in L$, $\overline{co}f^{-1}(x) = f^{-1}(x)$, *for every* $x \in f(L)$ *and* $f(L) = \overline{co}\, f(L)$ *be compact. Then there exists* $x_o \in L$ *such that* $x_o \in f(x_o)$.

Applying Theorem B we shall prove the following coincidence point theorem.

THEOREM 2. *Let* S *be a nonempty, compact and convex subset of Zima's type of tolopogical vector space* E, K *a compact subset of* E *such that* $S \subseteq K \subseteq E$, H *a convex subset of a topological vector space*, $\phi:S \to 2^H_c$, $\psi:K \to 2^H_{co}$ *(all convex subsets of* H*) upper semicontinuous mappings such that for every* $x \in S$:

$$\phi(x) \cap \psi(K) \neq \emptyset, \qquad \psi(x) \cap \phi(S) \neq \emptyset$$

and $\psi^{-1}(\phi(S)) = \overline{co}\psi^{-1}(\phi(S))$. *If for every* $x_1, x_2 \in S$ *and every* $\alpha, \beta \geq 0$, $\alpha+\beta = 1$, $\alpha\phi(x_1) + \beta\phi(x_2) \subseteq \phi(\alpha x_1+\beta x_2)$ *it follows that there exists* $x_o \in S$ *so that* $\phi(x_o) \cap \psi(x_o) \neq \emptyset$.

P r o o f. Let, as in Theorem 1:

$$\hat{\phi}(x) = \psi^{-1}(\phi(x)), \quad x \in S .$$

From the condition $\phi(x) \cap \psi(K) \neq \emptyset$, for every $x \in S$ it follows that $\hat{\phi}(x) \neq \emptyset$ for every $x \in S$. It is obvious that $\hat{\phi}(S) \subseteq K$. Let us prove that all the conditions of Theorem B for $L = S$ are satisfied.

First, let us prove that $L \subseteq \hat{\phi}(L)$. Since:

$$\psi^{-1}(\phi(S)) = \{x \mid x \in K, \ \psi(x) \cap \phi(S) \neq \emptyset\}$$

from $\psi(x) \cap \phi(S) \neq \emptyset$ for every $x \in S$, it follows that $S \subseteq \psi^{-1}(\phi(S))$ and so $L \subseteq \hat{\phi}(L)$. Furthermore, $\phi(x)$ is closed and ψ is upper semicontinuous and so $\hat{\phi}(x)$ is closed for every $x \in S$. It is obvious that $\hat{\phi}(S) = \overline{co}\hat{\phi}(S)$ and so it remains to be proved that:

$$\overline{co}\hat{\phi}^{-1}(x) = \hat{\phi}^{-1}(x), \quad \text{for every} \quad x \in \hat{\phi}(S)$$

and that $\hat{\phi}$ is upper semicontinuous. The upper semicontinuity can be proved as in [15]. Since E is Hausdorff, $\{x\}$ closed and $\hat{\phi}$ upper semicontinuous, we conclude that $\hat{\phi}^{-1}(\{x\})$ is closed. We shall prove the convexity of $\hat{\phi}^{-1}(x)$, $x \in \hat{\phi}(S)$. Let $u_1, u_2 \in \hat{\phi}^{-1}(x)$, $\alpha, \beta \geq 0$, $\alpha+\beta = 1$. This means that $\phi(u_1) \cap \psi(x) \neq \emptyset$, $\phi(u_2) \cap \psi(x) \neq \emptyset$. If $y_1 \in \phi(u_1) \cap \psi(x)$ and $y_2 \in \phi(u_2) \cap \psi(x)$, then $\alpha y_1 + \beta y_2 \in \psi(x)$, since $\psi(x)$ is convex and $\alpha y_1 + \beta y_2 \in \alpha\phi(u_1) + \beta\phi(u_2) \subset \phi(\alpha u_1 + \beta u_2)$.

So, we conclude that $\alpha y_1 + \beta y_2 \in \phi(\alpha u_1 + \beta u_2) \cap \psi(x)$.

3. EQUILIBRIUM STATE IN A SPECIAL NONCOOPERATIVE GAME

In this section we shall use Theorem A in order to obtain, similarly as in [12], a theorem on the equilibrium state in a special noncooperative game.

THEOREM 3. *Let $\{E_i\}_{i \in I}$ be a family of topological vector spaces, for each $i \in I$, K_i a closed and convex subset of*

E_i, $K = \prod_{i \in I} K_i$, $E = \prod_{i \in I} E_i$, *for each* $i \in I$, $\phi_i : K \to R(K_i)$ *be upper semicontinuous and* $\phi_i(K) \subseteq C_i \subseteq K_i$ $(i \in I)$ *where* C_i *is a compact set for each* $i \in I$. *If, for every* $i \in I$, $\phi_i(K)$ *is of Zima's type, there exists an* $\bar{x} \in K$ *such that* $\bar{x}_i \in \phi_i(\bar{x})$ *where* $\bar{x}_i = \text{proj}_{K_i} \bar{x}$, $i \in I$.

P r o o f. The proof is similar to [12]. Let $\phi : K \to R(K)$ be defined by:

$$\phi(x) = \prod_{i \in I} \phi_i(x), \quad x \in K .$$

Then, $\phi(x)$ is compact, since it is the product of compact sets $\phi_i(x)$, $(i \in I)$. Furthermore, $\phi(x)$ is convex and so $\phi(x) \in R(K)$ for every $x \in K$. Since $\phi(K) \subseteq \prod_{i \in I} C_i$, the upper semicontinuity of ϕ follows from the closedness of the graph of ϕ. This can be proved as in [15]. It remains to be proved that $\phi(K)$ is of Zima's type. Let us denote by V the fundamental system of neighborhoods of zero in E and by V_i the fundamental system of neighbourhoods of zero in E_i $(i \in I)$. Let $V \in V$. We shall show that there exists $U \in U$ so that $co(U \cap (\phi(K) - \phi(K))) \subseteq V$. Since $V \in V$, there exists a finite set $J \subsetneq I$ such that $V = \prod_{i \in I} V_i'$

where $$V_i' = \begin{cases} E_i, & i \in I \setminus J \\ V_i, & i \in J \end{cases}$$ and $V_i \in V_i$, $i \in J$.

Since $\phi_i(K)$ is of Zima's type, there exists $U_i \in V_i$ $(i \in I)$ so that:

$$co(U_i \cap (\phi_i(K) - \phi_i(K))) \subseteq V_i, \quad i \in J .$$

Let $$U_i' = \begin{cases} U_i, & i \in J \\ E_i, & i \in I \setminus J \end{cases}$$ $(i \in I)$. Then it is easy to prove [12] that for $U = \prod_{i \in I} U_i'$ we have that $co(U \cap (\phi(K) - \phi(K))) \subseteq V$.

As in [15] we can formulate the following theorem on the equilibrium state in a special noncooperative game.

THEOREM 4. *Let* $\{E_i\}_{i\in I}$ *be a family of topological vector spaces, for each* $i \in I$ *let* C_i *be a nonempty convex subset of* E_i, $C = \prod_{i\in I} C_i$, $\phi_i : C \to 2^{K_i}$ *be a continuous function for every* $i \in I$, *where* K_i *is a nonempty compact subset of* C_i *and* $f_i : C \to R^1$ *be a continuous function for every* $i \in I$. *If the set:*

$$(3)\qquad \hat{\phi}_i(x) = \{y \mid y \in \phi_i(x), \qquad f_i(y, x_i') = \max_{\hat{x}_i \in \phi_i(x)} f_i(\hat{x}_i, x_i')\}$$

is convex for each $x \in C$ *where,* $x_i' = \mathrm{proj}_{C_i'} x$, $C_i' = \prod_{j\in I, j\neq i} C_j$ $(i \in I)$ *and* $\phi_i(C)$ *is of Zima's type, then there exists an* $\bar{x} \in K = \prod_{i\in I} K_i$ *such that:*

$$(4)\qquad f_i(\bar{x}) = \max_{\hat{x}_i \in \phi_i(\bar{x})} f_i(\hat{x}_i, \bar{x}_i') \text{ and } \bar{x}_i \in \phi_i(\bar{x}) .$$

P r o o f. The proof follows from Lemma 5 in [15] and Theorem 3.

COROLLARY 2. *Let* $\{(E_i, \|\ \|_i^*)\}_{i\in I}$ *be a family of paranormed spaces,* $C = \prod_{i\in I} C_i$, $\phi_i : C \to 2^{K_i}$ *be a continuous function for every* $i \in I$ *where* K_i *is a nonempty compact subset of nonempty convex subset* C_i *of* E_i, $f_i : C \to R^1$ *be a continuous function for every* $i \in I$ *so that* $\hat{\phi}_i(x)$ $(x \in C)$, *defined by (3), is convex. If there exists* M_i $(i \in I)$ *so that:*

$$\|tx\|_i^* \le M_i t \|x\|_i^* , \quad t \in [0,1], \ x \in \phi_i(C) - \phi_i(C) ,$$

then there exists an $\bar{x} \in K = \prod_{i\in I} K_i$ *such that (4) holds.*

4. EXISTENCE THEOREMS FOR SOME CLASSES OF EQUATIONS

Using Corollary 1 we shall prove the following theorem.

THEOREM 5. *Let* X *be a topological vector space,* $\mathcal{U}$ *the fundamental system of zero in* X*,* K *a compact and convex subset of* X*,* G *a linear one to one mapping from* X *into* X *such that* G *and* G^{-1} *is continuous,* $T \in L(X,X)$ *and* S *an upper semicontinuous mapping from* K *into* $\mathcal{R}(X)$ *such that the following two conditions are satisfied:*

(i) *For every* $y \in \overline{co}(S(K)-S(K))$ *there exists a unique* $x(y) \in G(K)$ *such that* $x(y) = Tx(y) + y$.

(ii) $0 \in G(K) \cap S(K)$ *and for every* $V \in \mathcal{U}$ *there exists* $U \in \mathcal{U}$ *so that* $co(U \cap (S(K)-S(K))) \subseteq V$.

Then there exists $x \in K$ *so that* $G(x) \in TG(x) + Sx$.

P r o o f. Let us define the mapping $R: \overline{co}(S(K)-S(K)) \to G(K)$ in the following way:

$$Ry = TRy + y, \text{ for every } y \in \overline{co}(S(K)-S(K)) .$$

We shall prove that the mapping R is continuous. Let $\{y_\alpha\}_{\alpha \in A}$ be a convergent net from $\overline{co}(S(K)-S(K))$ and $\lim_{\alpha \in A} y_\alpha = y \in \overline{co}(S(K)-S(K))$. Since $\{Ry_\alpha\}_{\alpha \in A} \subseteq G(K)$ and $G(K)$ is compact, there exists a subnet $\{y_{\alpha_\beta}\}_{\beta \in B}$ such that $z = \lim_{\beta \in B} Ry_{\alpha_\beta}$. Then from $Ry_{\alpha_\beta} = TRy_{\alpha_\beta} + y_{\alpha_\beta}$ we have $\lim_\beta Ry_{\alpha_\beta} = \lim_\beta TRy_{\alpha_\beta} + \lim_\beta y_{\alpha_\beta}$ and so $z = Ry$. From this it is easy to conclude that $Ry = \lim_\beta Ry_{\alpha_\beta}$. Furthermore, from $M \subseteq S(K)-S(K)$ it follows that $co\,R(M) = R(co\,M)$. Indeed, if $u \in coR(M)$, then:

$$u = \sum_{i=1}^{n} t_i u_i,\ u_i \in R(M),\ t_i \geq 0\ (i \in \{1,2,\dots,n\}),\ \sum_{i=1}^{n} t_i = 1.$$

Since $u_i \in R(M)$ $(i \in \{1,2,\dots,n\})$, there exists $v_i \in M$ $(i \in \{1, 2,\dots,n\})$ such that $u_i = Rv_i$ $(i \in \{1,2,\dots,n\})$ and we have that:

$$Rv_i = TRv_i + v_i (i \in \{1,2,\dots,n\}). \quad \text{Hence:}$$

$$\sum_{i=1}^{n} t_i Rv_i = T(\sum_{i=1}^{n} t_i Rv_i) + \sum_{i=1}^{n} t_i v_i, \quad \sum_{i=1}^{n} t_i v_i \in \text{co}\, M$$

which implies that $R(\sum_{i=1}^{n} t_i v_i) = \sum_{i=1}^{n} t_i Rv_i$. So, we have that $R(\text{co}\, M) = \text{co}\, R(M)$.

Since $0 \in S(K)$, it follows that $S(K) \subseteq \overline{\text{co}}(S(K)-S(K))$ and so we can define the mapping R^* in the following way: $R^*x = \bigcup_{y \in Sx} Ry$, for every $x \in K$. Since T is a linear mapping from X into X and for every $y \in \overline{\text{co}}(S(K)-S(K))$ there exists one and only one element $x(y) \in G(K)$ such that $Ry = TRy + y$, it follows that $R(0) = 0$.

So, for every $V \in \mathcal{U}$ there exists $V' \in \mathcal{U}$ such that:

$$R(V' \cap \text{co}(S(K)-S(K))) \subseteq V .$$

The rest of the proof is similar to [3], but we shall repeat it here for completeness. Namely, we shall prove that the mappings G and R^* satisfy all the conditions of Corollary 1 and that there exists $x \in K$ such that $G(x) \in R^*(x)$.

It is obvious that R^* is an upper semicontinuous mapping from K into $\mathcal{R}(X)$, since R is continuous and S is an upper semicontinuous mapping from K into $\mathcal{R}(X)$ and $M = S(x)$, $x \in K$ implies that $R(M)$ is convex. Furthermore, there exists $U' \in \mathcal{V}$ such that:

$$\text{co}(U' \cap (S(K)-S(K))) \subseteq V' \cap \text{co}(S(K)-S(K)) .$$

Since $R(\text{co}(U' \cap (S(K)-S(K)))) = \text{co}(R(U' \cap (S(K)-S(K))))$ it follows that $\text{co}(R(U' \cap (S(K)-S(K)))) \subseteq V$. We have that $R^{-1}z = z - Tz$ for every $z \in R(\overline{\text{co}}(S(K)-S(K)))$. Hence, R^{-1} is continuous, and so there exists $U \in \mathcal{U}$ such that:

$$R^{-1}(U \cap R(S(K)-S(K))) \subseteq U' \cap (S(K)-S(K)) .$$

Hence, $co(U \cap (R^*K-R^*K)) \subseteq V$, since for every $x,y \in S(K)$, $R(x-y) = Rx-Ry$. From Corollary 1 it follows that there exists $x \in K$ such that $G(x) \in R^*(x)$.

This means that there exists $u \in S(x)$ such that $G(x) = Ru$. So, we have that $G(x) = Ru = TRu + u = TG(x) + u \in TG(x) + S(x)$.

REMARK. From the proof it is easy to conclude that it is sufficient to suppose that K is a nonempty closed and convex subset for X, that the set $\overline{\{x(y)\}}_{y \in \overline{co}(S(K)-S(K))}$ is compact and S such that the set $\overline{S(K)}$ is compact.

That is, in this case we can also prove that the mapping R is continuous.

COROLLARY 3. [3] *Let X be a topological vector space, K a compact, convex subset of X,* $0 \in K$*,* $T \in L(X,X)$ *and* $S: K \to \mathcal{R}(X)$ *an upper semicontinuous mapping. Suppose that the following two conditions are satisfied:*

(i) *For every* $y \in \overline{co}(S(K)-S(K))$ *there exists one and only one element* $x(y) \in K$ *such that* $x(y) = Tx(y) + y$.

(ii) $0 \in S(K)$ *and* $S(K)$ *is of Zima's type.*

Then there exists $x \in K$ *such that* $x \in Tx + Sx$.

P r o o f. It is enough to take in Theorem 5 that $Gx = x$, $x \in X$.

COROLLARY 4. *Let* $(X, \|\ \|^*)$ *be a complete paranormed space, K a closed and convex subset of X,* $S: K \to \mathcal{R}(X)$ *an upper semicontinuous mapping,* $T \in L(X,X)$, $\overline{S(K)}$ *compact,* $0 \in S(K) \cap K$ *so that the following conditions are satisfied:*

1. $\|Tx\|^* \le q\|x\|^*$, *for every* $x \in X$, *where* $q \in [0,1)$
2. $T(K) + \overline{co}(S(K)-S(K)) \subseteq K$
3. *There exists* $C > 0$ *so that* $\|tz\|^* \le Ct\|z\|^*$ *for every* $t \in [0,1]$ *and every* $z \in S(K)-S(K)$.

Then, there exists $x \in K$ *such that* $x \in Tx + Sx$.

P r o o f. It remains to be proved that for every $y \in \overline{co}(S(K)-S(K))$ there exists one and only one element $x(y) \in K$ such that $x(y) = Tx(y) + y$ and the set $\{\overline{x(y)}\}_{y \in co(S(K)-S(K))}$ is compact. Since $\| Tx-Ty\|^* \le q\| x-y\|^*$, for every $x,y \in X$ and

$$T(K) + \overline{co}(S(K)-S(K)) \subseteq K$$

it follows from the Banach fixed point theorem that for every $y \in \overline{co}(S(K)-S(K))$ there exists $x(y) \in K$ so that $x(y) = Tx(y) + y$. From the inequality:

$$\| x(y_1) - x(y_2)\|^* \le \frac{\| y_1 - y_2\|^*}{1-q}$$

for every $y_1, y_2 \in \overline{co}(S(K)-S(K))$ follows the continuity of the mapping $y \mapsto x(y)$ $(y \in \overline{co}(S(K)-S(K)))$. The rest of the proof is similar to the proof of Theorem 5.

THEOREM 6. *Let* K *be a nonempty, compact, convex subset of topological vector space* E, F *a topological vector space,* g *a continuous mapping of* $K \times K$ *into* F, *and* C *a closed subset of* F. *Suppose that for each* x *in* K *the set:*

$$\{y \mid y \in K,\ g(x,y) \in C\}$$

is nonempty and convex. If K *is of Zima's type, there exists an element* $u \in K$ *such that* $g(u,u) \in C$.

P r o o f. As in [1] for each x in K, we define T(x) in the following way:

$$Tx = \{y \mid y \in K,\ g(x,y) \in C\}.$$

Since $T(K) \subseteq K$ and K is of Zima's type, it follows that T(K) is of Zima's type. Furthermore, the mapping $T: K \to 2^K$ satisfies the condition $T(x) = \overline{co}T(x)$ for every $x \in K$, since C is closed and g continuous. The upper semicontinuity of T follows as in [1], and so from Theorem A it follows that there exists $x \in K$ such that $x \in Tx$. This implies that $g(x,x) \in C$.

As in [1], from Theorem 6 we obtain the following Corollaries.

COROLLARY 5. *Let* K *be a compact, convex subset of topological vector space* E, F *a topological vector space, and* g *a continuous mapping of* K x K *into* F. *Suppose that* $g(x,t_1y_1 + t_2y_2) = t_1g(x,y_1)+t_2g(x,y_2)$ *for all* $x,y_1,y_2 \in K$ *and* $t_1,t_2 \geq 0$, $t_1 + t_2 = 1$ *and there exists for every* $x \in K$, $y \in K$ *such that* $g(x,y) = 0$. *If* K *is of Zima's type, there exists* $u \in K$ *such that* $g(u,u) = 0$.

P r o o f. For $C = \{0\}$, the set $\{y | y \in K,\ g(x,y) \in C\}$ is nonempty and convex for every $x \in K$.

COROLLARY 6. *Let* K *be a compact, convex subset of Zima's type of topological vector space* E, C *a nonempty, closed, convex subset of* E *such that* $f(K) \subseteq K+C$. *Then there exists an element* $u \in K$ *such that* $f(u) \in u+C$.

P r o o f. Let in Theorem 6, $F = E$ and $g(x,y) = f(x)-y$ for every $x,y \in K$. Then

$$\{y|y \in K,\ g(x,y) \in C\} = K \cap (f(x)-C)$$

and so the set $\{y|y \in K,\ g(x,y) \in C\}$ is nonempty and convex for every $x \in K$.

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REZIME

NEKE PRIMENE TEOREME O NEPOKRETNOJ TAČKI ZA VIŠEZNAČNA PRESLIKAVANJA U VEKTORSKO TOPOLOŠKIM PROSTORIMA

U ovom radu su dokazane teoreme o koincidenciji za višeznačna preslikavanja a data je i primena teoreme o nepokretnoj tački iz rada [3] u teoriji igara. Dokazane su i tri teoreme o postojanju rešenja nekih klasa jednačina u vektorsko topološkom prostoru.