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ON A COMMON FIXED POINT IN QUASI-UNIFORMIZABLE SPACES

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ABSTRACT

Using a similar method as in [7] we prove in this paper a generalization of Fisher's fixed point theorem in quasi-uniformizable spaces. First, we shall give some difinitions from [5] and [7].

1. Let X be an arbitrary set, $\{d_i | i\in I\}$ be a family of mappings of XxX into $\mathbb{R}^+$ and $g:I\to I$.

DEFINITION 1. *A triplet* $(X,\{d_i\}_{i\in I},g)$ *is said to be a quasiuniformizable space if for every* $x,y,z\in X$ *and* $i\in I$ *we have:*

(a) $d_i(x,y)\geq 0, d_i(x,x)=0$,

(b) $d_i(x,y)=d_i(y,x)$,

(c) $d_i(x,y)\leq d_{g(i)}(x,z)+d_{g(i)}(z,y)$.

A quasi-uniformizable space $(X,\{d_i\}_{i\in I},g)$ is Hausdorff if the relation $d_i(x,y)=0$, for every $i\in I$ implies $x=y$. A Hausdorff quasi-uniformizable space $(X,\{d_i\}_{i\in I},g)$ becomes a Hausdorff topological space if the fundamental system of neighbourhoods of $x\in X$ is given by:

$$B(x;\varepsilon,i)=\{y\in X : d_i(x,y)<\varepsilon\}.$$

A mapping $t: [0,1]^2\to[0,1]$ is called a T-norm if for every $a,b,c,d\in[0,1]$:

1. $t(0,0)=0, t(a,1)=a$,
2. $t(a,b)=t(b,a)$,

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3. $t(a,b) \geq t(c,d)$ if $a \geq c$ and $b \geq d$,

4. $t(t(a,b),c) = t(a,t(b,c))$.

Now, let X be a vector space. I be an index set, L be a family of distribution functions and for every $i \in I$, $F^i : X \to L$.

DEFINITION 2. [5] *A triplet* $(X,\{F^i\}_{i\in I},t)$ *is called a probabilistic locally convex space if* t *is a* T-*norm and for every* $i\in I$ *the following conditions are satisfied* ($F^i_x = F^i(x)$):

A. $F^i_x(s) = 1$, *for every* $s > 0 \Longleftrightarrow x=0$,

B. $F^i_x(0)=0$, *for every* $x\in X$,

C. $F^i_{rx}(s) = F^i_x\left(\frac{s}{|r|}\right)$, *for every* $x\in X, s>0$ *and* $r \in K\setminus\{0\}$ (K *is the scalar field).*

D. $F^i_{x+y}(s_1+s_2) \geq t(F^i_x(s_1), F^i_y(s_2))$, *for every* $x,y\in X$ *and every* $s_1,s_2>0$.

In X the (ε,λ)-topology is introduced in the following way:

The fundamental system of neighbourhoods of $x\in X$ is given by the family $\mathcal{U}=\{U^i_x(\varepsilon,\lambda): i\in I,\ \varepsilon>0, \lambda\in(0,1)\}$ where $U^i_x(\varepsilon,\lambda) = \{y\in X: F^i_{x-y}(\varepsilon) > 1-\lambda\}$.

In [7] it is shown that a probabilistic locally convex space $(X,\{F^i\}_{i\in I},t)$, such that $\sup_{a<1} t(a,a)=1$, is a quasiuniformizable space in which the family $\{d_i\}_{i\in I'}$ is defined in the following way:

For $j=(i,\lambda)\in I'$, where $i\in I$ and $\lambda\in(0,1)$, $d_j(x,y) = \sup\{s: F^i_{x-y}(s) \leq 1-\lambda\}$.

The construction of the mapping $g: I' \to I'$ is as follows. From $\sup_{a<1} t(a,a)=1$ it follows that for every $\lambda\in(0,1)$ the exists $\delta_\lambda \in (0,1)$ so that for every $\delta \leq \delta_\lambda$, $t(1-\delta,1-\delta) \geq 1-\frac{\lambda}{2}$.

Let $\bar{g}(\lambda) = \sup\{\delta_\lambda$: where δ_λ is defined above $\}$.

Then $g(j) = (i,\bar{g}(\lambda))$, for $j=(i,\lambda)$.

2. Now, we shall give a generalization of a common fixed point theorem from [1] in quasi-uniformizable spaces. This theorem is also a generalization of Theorem 1 from [4].

THEOREM *Let* $(X,\{d_i\}_{i\in I},g)$ *be a sequentially complete Hausdorff quasi-uniformizable space,* $f:I\to I$, S *and* T *be continuous mappings from* X *into* X, $A:X\to SX\cap TX$ *be continuous so that* A *commutes with* S *and* T *and the following conditions are satisfied:*

1. *For every* $i\in I$, *there exists* $q_i:\mathbb{R}^+\to[0,1]$, *which is a nondecreasing function for which is* $\overline{\lim\limits_{n\to\infty}}\, q_{f^n(i)}(t)<1$ *for every* $i\in I$ *and every* $t\in\mathbb{R}^+$ *and:*

   $d_i(Ax,Ay)\le q_i(d_{f(i)}(Sx,Ty))d_{f(i)}(Sx,Ty)$

   for every $i\in I$ *and every* $x,y\in X$.
2. *There exists* $x_o\in X$ *so that for every* $i\in I$:

   $$\sup_{j\in O(i,f),\,p\in N} d_j(Ax_o,Ax_p)=K_i\in\mathbb{R}^+$$

   where $O(i,f)=\{i,f(i),f^2(i),\dots\}$ *and* $\{x_p\}_{p\in N}$ *is defined by:* $Sx_{2n-1}=Ax_{2n-2}$, $Ax_{2n-1}=Tx_{2n}$ $(n\in N)$.

Then there exists $z\in X$ *so that* $Az=Sz=Tz$. *If, in addition, for every* $i\in I$:

$$\text{(1)}\quad \sup_{j\in O(i,f)} d_j(A^3x_1,A^2x_o)=M_i\in\mathbb{R}^+$$

then Az *is a common fixed point for* A,S *and* T. *Further, let* $M=\{w: w\in X, w=Aw=Sw=Tw$, *there exists* $\{R_i\}_{i\in I}$, *so that for every* $i\in I: \sup_{j\in O(i,f)} d_j(Az,w)\le R_i\}$. *Then* $M=\{Az\}$.

Proof: Similarly as in [4] it is easy to prove that for every $k\in N$ and $i\in I$:

$$d_i(Ax_{2k},Ax_{2k-1}) \le \prod_{s=0}^{2k-2} q_{f^s(i)}(K_i)K_i$$

$$d_i(Ax_{2k+1},Ax_{2k}) \le \prod_{s=0}^{2k-1} q_{f^s(i)}(K_i)K_i \, .$$

Since $\overline{\lim\limits_{n\to\infty}}\, q_{f^n(i)}(K_i)\le Q_i<1$, for every $i\in I$, it follows that there exists $n_i \in N$ so that $q_{f^n(i)}(K_i)\le Q_i$, for every $n\ge n_i$ which implies that:

$$d_i(Ax_n,Ax_{n-1})\le S_i Q_i^n \ , \text{ for every } i\in I, \quad n\in N.$$

Let us prove that $\{Ax_n\}_{n\in N}$ is a Cauchy sequence which means that for every $i\in I$ and every $\varepsilon>0$ there exists $n(i,\varepsilon)\in N$ so that $d_i(Ax_n,Ax_{n+p})<\varepsilon$, for every $n\ge n(i,\varepsilon)$, $p\in N$. Let $m\ge k$, $i\in I$. From the definition of the sequence $\{x_n\}_{n\in N}$ and condition 1 of the Theorem it follows that:

$$d_i(Ax_{2k},Ax_{2m+1})\le q_i(d_{f(i)}(Ax_{2m},Ax_{2k-1}))\dots q_{f^{2k-1}(i)}($$

$$(d_{f^{2k}(i)}(Ax_0,Ax_{2m+1-2k}))d_{f^{2k}(i)}(Ax_0,Ax_{2m+1-2k}))$$

and similarly for $2k>2m+1$:

$$d_i(Ax_{2k},Ax_{2m+1})\le q_i(d_{f(i)}(Ax_{2m},Ax_{2k-1}))\dots q_{f^{2m}(i)}($$

$$(d_{f^{2m+1}(i)}(Ax_0,Ax_{2k-2m-1}))d_{f^{2m+1}(i)}(Ax_0,Ax_{2k-2m-1})).$$

Using condition 2 and the property of q_i that $q_i(t)\le 1$ for every $t\in \mathbb{R}^+$ we obtain that for every $i\in I$:

$$d_i(Ax_{2k},Ax_{2m+1})\le q_i(K_i)\dots q_{f^{2k-1}(i)}(K_i)K_i \ (m\ge k)$$

and :

$$d_i(Ax_{2k},Ax_{2m+1})\le q_i(K_i)\dots q_{f^{2m}(i)}(K_i)K_i \ (2k > 2m+1).$$

This implies that :

$$d_i(Ax_n,Ax_{n+p})\le q_i(K_i)\dots q_{f^{n-1}(i)}(K_i)K_i$$

for every $i \in I$ and for $n=2k, p=2m+1$ or $n=2k+1$, $p=2m+1$. Let $p=2m$ and $n=2k$ or $n=2k+1$. Then:

$$d_i(Ax_n,Ax_{n+p}) \le d_{g(i)}(Ax_n,Ax_{n+1})+d_{g(i)}(Ax_{n+1},Ax_{n+1+p-1}) \le$$

$$\le \prod_{s=0}^{n-1} q_{f^s(g(i))}(K_{g(i)})K_{g(i)} + \prod_{s=0}^{n} q_{f^s(g(i))}(K_{g(i)})K_{g(i)}.$$

Since $\overline{\lim}_{s\to\infty} q_{f^s(g(i))}(K_{g(i)})<1$, for every $i \in I$ it follows that $\{Ax_n\}_{n\in N}$ is a Cauchy sequence. Let $z=\lim_{n\to\infty} Ax_n$. As in [4] it follows that $Az=Sz=Tz$.

Further (1) implies that Az is a common fixed point for A,S and T. Indeed, from $d_j(A^2z,Az)=\lim_{n\to\infty} d_j(A^2Ax_{2n+1},AAx_{2n})$ for every $j \in O(i,f)$. using condition 1.and (1) we conclude that $d_j(A^2z,Az)\le M_i$, for every $j \in O(i,f)$, since for every $n \in N$:

$$d_j(A^3x_{2n+1},A^2x_{2n}) \le d_{f^{2n}(j)}(A^3x_1,A^2x_o) \le M_i, \text{ for every } i \in I,$$

where $j \in O(i,f)$.

So we have that:

$$d_i(A^2z,Az) \le q_i(M_i)q_{f(i)}(M_i)\dots q_{f^n(i)}(M_i)M_i$$

for every $i \in I$ which implies that $d_i(A^2z,Az)=0$, for every $i \in I$, This implies that Az is a common fixed point for A,S and T.

Let us prove that $M=\{Az\}$. Suppose that $w=Aw=Tw=Sw$ and for every $i \in I$:

$$\sup_{j\in O(i,f)} d_j(Az,w)\le R_i .$$

Then :

$$d_i(Az,w) = d_i(A(Az),Aw)\le q_i(d_{f(i)}(S(Az),Tw))d_{f(i)}(S(Az),Tw)=$$

$$=q_i(d_{f(i)}(Az,w))d_{f(i)}(Az,w)\le\dots\le q_i(d_{f(i)}(Az,w))\dots \times$$

$$q_{f^n(i)}(d_{f^{n+1}(i)}(Az,w))d_{f^{n+1}(i)}(Az,w).$$

From this it follows that $d_j(Az,w)\le R_i$ $(j \in O(i,f))$ and so $d_i(Az,w)\le q_i(R_i)q_{f(i)}(R_i)\dots q_{f^n(i)}(R_i)R_i$.

Since $\overline{\lim_{n\to\infty}}\, q_{f^n(i)}(R_i)<1$ it follows that $d_i(Az,w)=0$, for every $i\in I$ and so $Az=w$.

Remark: If there exists $u\in X$ so that for every $i\in I$:

$$d_j(Az,u)\le T_i,\ \text{for every } j\in O(i,f)$$

and $g:O(i,f)\to O(i,f)$, for every $i\in I$, then there exists one and only one common fixed point $w\in X$ for A,S and T so that for every $i\in I$:

$$d_j(w,u)\le T_i'\ ,\ \text{for every } j\in O(i,f).$$

Namely, then we have :

$$d_j(Az,w)\le d_{g(i)}(Az,u)+d_{g(i)}(u,w)\le T_i+T_i'\ ,\ \text{for every}$$

$i\in I$ and every $j\in O(i,f)$ and in the Theorem is proved that $Az=w$.

COROLLARY 1. *Let* $(X,\{d_i\}_{i\in I})$ *be a sequentially complete Hausdorff uniformizable space,* $f:I\to I$, S *and* T *be continuous mappings from* X *into* $X, A:X\to SX\cap TX$ *be continuous so that condition 1. of the Theorem is satisfied and there exists* $x_o\in X$ *and* $x_1\in X$ *so that* $Sx_1=Ax_o$ *and for every* $i\in I$:

$$\sup_{n\in N} d_{f^n(i)}(Ax_o,Ax_1)=K_i,\ K_i\in \mathbb{R}^+$$

Then there exists $z\in X$ *so that* $Az=Sz=Tz$. *If, in addition, for every* $i\in I$: $\sup_{n\in N} d_{f^n(i)}(A^3x_1,A^2x_o)=M_i$, $M_i\in\mathbb{R}^+$ *then* Az *is a common fixed point for* A,S *and* T. *Further, if for every* $i\in I$:

$$\sup_{n\in N} d_{f^n(i)}(A^2x_1,A^2x_o)=R_i,\ R_i\in\mathbb{R}^+ \tag{2}$$

then there exists one and only one element $w\in X$ *such that*

(3) $\sup_{n\in N} d_{f^n(i)}(w,A^2x_o)=N_i, N_i \in \mathbb{R}^+$ *, for every* $i\in I$

and $Aw = Sw = Tw = w$.

P r o o f: Every uniformizable space $(X,\{d_i\}_{i\in I})$ is a quasiuniformizable space, where $g(i)=i$, for every $i\in I$. So we have that:

$$d_i(Ax_p,Ax_o)\le d_i(Ax_p,Ax_{p-1})+d_i(Ax_{p-1},Ax_{p-2})+\ldots+d_i(Ax_1,Ax_o)$$

(for every $i\in I, p\ge 2$). Since for every $j\in O(i,f)$

$$d_j(Ax_p,Ax_{p-1})\le \prod_{s=0}^{p-2} q_{f^s(j)}(K_i)K_i,$$ for every $i\in I$, it follows that :

$$d_j(Ax_p,Ax_o)\le\sum_{r=1}^{p}\left(\prod_{s=0}^{r-2} q_{f^s(j)}(K_i)\right)K_i.$$

Since $\overline{\lim_{n\in N}}\, q_{f^n(i)}(K_i)<1$ there exists $n_i\in N$ so that :

$$q_{f^n(i)}(K_i)\le Q_i<1, \text{ for every } n\ge n_i$$

and if $j\in\{f^s(i)\mid s\ge n_i\}$ then:

$$d_j(Ax_p,Ax_o)\le\sum_{r=1}^{\infty} Q_i^{r-1}K_i, \text{ for every } p\ge 2.$$

Since $Q_i<1$ it is easy to see that condition 2.of the Theorem is satisfied. So, there exists $z\in X$ such that Az is a common fixed point for A,S and T. From (2) it follows that $d_j(Az,A^2x_o)\le R_i$, for every $i\in I$ and every $j\in O(i,f)$ and using the Remark we conclude that there exists one and only one element $w\in X$ such that w is a common fixed point for A, S and T and that (3) is satisfied.

Using the Theorem we obtain the following corollory which is a generalization of the Theorem 1 from [2] .

COROLLARY 2 *Let* $(X,\{F^i\}_{i\in I},t)$ *be a sequentially complete Hausdorff probabilistic locally convex space where*

$\sup_{a<1} t(a,a)=1$, $f:I\to I$, S *and* T *be continuous mappings from* X *into* X, $A:X\to SX\cap TX$ *be a continuous mapping which commutes with* S *and* T *and the following conditions are satisfied:*

(i) *For every* $i\in I$, *there exists* $q_i:\mathbb{R}^+\to[0,1]$, *which is a nondecreasing function continuous from the right such that for every* $i\in I$ *and every* $s\in\mathbb{R}^+$ $\overline{\lim_{n\to\infty}}\, q_{f^n(i)}(s)<1$ *and for every* $i\in I$, *every* $x,y\in X$ *and every* $s\in\mathbb{R}^+$:

$$F^i_{Ax-Ay}(q_i(s)s)\geq F^{f(i)}_{Sx-Ty}(s)$$

(ii) *There exists* $x_o\in X$ *so that for every* $i\in I$: $\overline{\lim_{s\to\infty}}\, F^j_{Ax_o-Ax_p}(s)=1$, *uniformly in* $j\in O(i,f)$ *and* $p\in N$, *where* $\{x_p\}_{p\in N}$ *is defined by* $Sx_{2n-1}=Ax_{2n-2}$, $Tx_{2n}=Ax_{2n-1}$ *for every* $n\in N$.

Then there exists $z\in X$ *so that* $Az=Sz=Tz$. *If, in addition, for every* $i\in I$, $\lim_{s\to\infty} F^j_{A^3x_1-A^2x_o}(s)=1$, *uniformly in* $j\in O(i,f)$ *then* Az *is a common fixed point for* A,S *and* T. *Further, let* $M'=\{w: w\in X, w=Aw=Sw=Tw$, *for every* $i\in I$, $\lim_{s\to\infty} F^j_{Az-w}(s)=1$, *uniformly in* $j\in O(i,f)\}$. *Then* $M'=\{Az\}$.

P r o o f: As in [7] it follows that (i) and (ii) implies 1. and 2. from the Theorem and that (1) is satisfied since for every $i\in I$, $\lim_{s\to\infty} F^j_{A^3x_1-A^2x_o}(s)=1$, uniformly in $j\in O(i,f)$, where $d_j(x,y)=\sup\{s: F^i_{x-y}(s)\leq 1-\alpha\}$, for every $j=(i,\alpha)\in I'$, $i\in I$ and $\alpha\in(0,1)$. Since, $w\in M'$ implies $w\in M$, where M is from the Theorem, it follows $M'=\{Az\}$.

Remark: If AX is a probabilistic bounded subset of $SX\cap TX$, (i) from Corollary 2 is satisfied and for every $i\in I$ there exists $h(i)\in I$ such that:

$$F^{f^n(i)}_x(s)\geq F^{h(i)}_x(s),\ \text{for every } s>0,\ \text{every } x\in X \text{ and}$$

every $n\in N$ it is easy to see that there exists one and only one element $x\in X$ such that Ax is the unique common fixed point for A,S and T.

REFERENCES

[1] B.Fisher, *Mappings with a Common Fixed Point, Math.Sem.Notes, 7,1979, 81-84*

[2] O.Hadžić, *A fixed point theorem in probabilistic locally convex spaces, Revue Roum. Math.Pure Appl. 23,5(1978), 735-744*

[3] O.Hadžić, *On Common Fixed Points in Probabilistic Metric Spaces, Math. Sem.Notes, 10(1982), 31-39*

[4] O.Hadžić, ***On Common Fixed Points in Uniformizable Spaces, Anal. Num.Appr., 12, 1(1983), 45-54***

[5] V.I.Istrăţescu, *Introducere in teoria spatiilor metrice probabiliste cu aplicatii, Editura Technica, Bucuresti, 1974*

[6] Duong Trong Nhan, *Pair of nonlinear contraction mappings **and common fixed** points, Studia Univ. Babes-Bolyai, Mathematica, XXVI,1, 1981. 34-51*

[7] Do Hong Tan, *On the contraction principle in uniformizable spaces, Acta Mathematica Vietnamica, Tom 5,No. 1980, 88-99*

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U ovom radu su uopšteni rezultati rada [4].