Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu knjiga 13 (1983)
Review of Research Faculty of Science-University of Novi Sad, Volume 13(1983)

ON THE NUMBER OF ABELIAN GROUPS OF A GIVEN ORDER AND THE NUMBER OF PRIME FACTORS OF AN INTEGER

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ABSTRACT

Let $a(n)$ and $\omega(n)$ denote the number of non-isomorphic abelian groups with n elements and the number of distinct prime factors of n respectively. The distribution of values of $a(n)$ (which is multiplicative) and $\omega(n)$ (which is additive) is compared in several ways.

1. INTRODUCTION

Let, as usual, $a(n)$ denote the number of non-isomorphic abelian groups with n elements. It is well-known (see [3]) that $a(n)$ is a multiplicative function of n such that $a(p^k) = P(k)$ for every prime p and every natural number k, where $P(k)$ is the number of unrestricted partitions of k (here and later $p, p_1, p_2, \ldots$ denote primes). Various problems concerning the distribution of values of $a(n)$ and related multiplicative functions were investigated in [4], [5] and [8]. Thus for

AMS Mathematics subject classification (1980): 10H25
Key words and phrases: Nonisomorphic abelian groups, asymptotic formulas, distribution values.

instance it was proved in [4] that for $k \geq 1$ fixed

$$A_k(x) = \sum_{n \leq x, a(n)=k} 1 = d_k x + O(x^{1/2}\log x) \tag{1.1}$$

holds uniformly in k ($f(x) = O(g(x))$ and $f(x) \ll g(x)$ both mean $|f(x)| < Cg(x)$ for $x \geq x_0$ and some constant $C > 0$). The non-negative constant d_k is called the local density of $a(n)$, and as shown in [5] it satisfies the inequality

$$d_k \leq c_1 \exp(-c_2 \log k \cdot \log\log k), \quad (k \geq 3) \tag{1.2}$$

with some $c_1, c_2 > 0$. From the product representation

$$\sum_{n=1}^{\infty} a(n) n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\zeta(4s)\ldots \quad (\mathrm{Re}\ s > 1) \tag{1.3}$$

it is seen that the mean value of $a(n)$ equals $\zeta(2)\zeta(3)\zeta(4)\ldots = 2.29485\ldots$. Thus $a(n)$ is small on the average, although one has (see [7])

$$\limsup_{n\to\infty} \log a(n) \log\log n/(\log n) = (\log 5)/4, \tag{1.4}$$

and the bound implied by (1.4) is asymptotically attained for $n = (p_1 p_2 \ldots p_k)^4$, where p_i is the i-th prime. It seemed interesting to compare the values of $a(n)$ and other common arithmetical functions such as $d(n)$ and $\omega(n)$, which represent the number of divisors and the number of distinct prime factors of n respectively. From the elementary formulas

$$\sum_{n \leq x} d(n) = x\log x + (2\gamma - 1)x + O(x^{1/2}) \quad (\gamma = 0.577\ldots) \tag{1.5}$$

and

$$\sum_{n \leq x} \omega(n) = x\log\log x + Bx + O(x/\log x), \quad (B > 0) \tag{1.6}$$

it is seen that the average order of $d(n)$ and $\omega(n)$ is $\log n$ and $\log\log n$ respectively.
Therefore it is no surprise that

$$\sum_{n\le x,\, d(n)>a(n)} 1 = x + O(x\log^{\varepsilon-1}x), \tag{1.7}$$

$$\sum_{n\le x,\, \omega(n)>a(n)} 1 = x + O(x(\log\log x)^{-K}). \tag{1.8}$$

These formulas were proved in [5], and here $0 < \varepsilon < 1$, while K is an arbitrary, but fixed positive number.

2. STATEMENT OF RESULTS

In this note we shall further compare the values of $a(n)$ and $\omega(n)$. Possible generalizations to other arithmetical functions which behave "similarly" as $a(n)$ and $\omega(n)$ will be omitted to make the exposition clearer; e.g. functions of class F_1 of [5] can be obviously considered instead of $a(n)$ only, and likewise instead of $\omega(n)$ one may consider the familiar function $\Omega(n)$, the number of all prime factors of n etc. The problem to compare the values of $a(n)$ and $\omega(n)$ seems interesting, because $a(n)$ is multiplicative ($a(mn) = a(m)a(n)$ for $(m,n) = 1$) and $\omega(n)$ is additive ($\omega(mn) = \omega(m) + \omega(n)$ for $(m,n) = 1$). In treating two multiplicative or two additive functions, one can make use of the fact that the product (or quotient) of two multiplicative functions is again multiplicative, while the sum (or difference) of two additive functions is again additive. However, in our case special methods have to be used which will simultaneously deal with $a(n)$ and $\omega(n)$. The first result is an improvement of (1.8), which we formulate as

THEOREM 1. *There is a constant* $C > 0$ *such that*

$$\sum_{n\le x,\, \omega(n)>a(n)} 1 = x + O(x\exp(-C\log_3 x\log_4 x)) . \tag{2.1}$$

Here we used the abbreviation $\log_r x = \log(\log_{r-1}x)$, $\log_1 x = \log x$ = the natural logarithm of x. At this point it may be remarked that the equation $a(n) = \omega(n)$ holds for many n, and quantitatively we have, for any fixed $A > 0$,

$$(2.2) \qquad \sum_{n \le x, \omega(n)=a(n)} 1 \gg x(\log\log x)^A/\log x \,.$$

To see this recall that for an infinity of integers $k \ge 5$ there exists an integer $r < k$ such that $P(r) = k$. Let $m = p_1p_2\cdots p_{k-1}p_k^r$, where the p_i's are distinct primes. Then $\omega(m) = k = P(r) = a(m)$, hence for k fixed

$$(2.3) \qquad \sum_{n \le x, \omega(n)=a(n)} 1 \ge \sum_{m \le x} 1 \gg x(\log\log x)^{k-2}/\log x,$$

by a classical result of E. Landau (see p. 168 of [6]) concerning the number of n not exceeding x for which $\omega(n) = k$. Now (2.2) follows from (2.3), since the values of the partition function tend quickly to infinity because

$$(2.4) \qquad P(k) = (1 + o(1))(4\sqrt{3}k)^{-1}\exp(\pi(2k/3)^{1/2}), \quad (k \to \infty)$$

by a classical result of G.H. Hardy and S. Ramanujan ([9], p. 240).

The next result shows that the equation $\omega(n) = ra(n)$ has many solutions for any real number $r > 0$. The result is

THEOREM 2. *Every real number* $r \ge 0$ *is the limit point of the sequence* $\omega(n)/a(n)$.

Finally we present an asymptotic formula for a sum involving the functions $\omega(n)$ and $a(n)$. This is

THEOREM 3. *There is a constant* $A > 0$ *such that*

$$(2.5) \qquad \sum_{n \le x} \left(\frac{\omega(n) - \log\log n}{a(n)}\right)^2 = Ax\log\log x + O(x) \,.$$

As a corollary it follows that for almost all n we have

$$(2.6) \qquad |\omega(n) - \log\log n| \le a(n)(\log\log n)^{1/2+\delta}$$

for any $0 < \delta < 1/2$. To see this, let $F(\delta,x)$ denote the number of $n \le x$ for which (2.6) fails to hold. Then, using (2.5), we obtain

$$F(\delta,x) \le \sum_{n\le x} \left(\frac{\omega(n) - \log\log n}{a(n)}\right)^2 (\log\log n)^{-1-2\delta} <<$$

$$x^{1/2+\varepsilon} + \sum_{\sqrt{x}<n\le x} \left(\frac{\omega(n) - \log\log n}{a(n)}\right)^2 (\log\log x)^{-1-2\delta} << x(\log\log x)^{-2\delta},$$

since $\log\log n = \log\log x + O(1)$ for $\sqrt{x} < n \le x$. The last expression above is $o(x)$ for $\delta > 0$ as $x \to \infty$, which justifies the claim that (2.6) holds for "almost all" n. Theorem 3 could be generalized by replacing $a(n)$ by $a^k(n)$ for any fixed integer $k \ge 1$, in which case the constant A in (2.5) would depend on k.

3. PROOFS OF THE THEOREMS

To prove (2.1) let $S(x)$ denote the number of $n \le x$ for which $\omega(n) \le a(n)$. Write

$$(3.1) \qquad S(x) = \sum_{n\le x,\,\omega(n)\le a(n)} 1 = S_1 + S_2$$

say, where in S_1 we sum over relevant n for which $a(n) \le \frac{1}{3}\log_2 x$, while in S_2 we sum over relevant n for which $a(n) > \frac{1}{3}\log_2 x$. Using a result of P. Erdös and J.-L. Nicolas [2] on the distribution of values of $\omega(n)$ we have

$$(3.2) \qquad S_1 \le \sum_{n\le x,\,\omega(n)\le(\log_2 x)/3} 1 << x\log^{-c} x$$

for some $0 < c < 1$ (the exact value of c is unimportant here). To bound S_2 we use (1.1), (1.2) and (1.4) to obtain

$$(3.3) \qquad S_2 \le \sum_{(\log_2 x)/3\le k\le \exp(2\log x/\log_2 x} (d_k x + O(x^{1/2}\log x))$$

$$\ll x \sum_{k \geq (\log_2 x)/3} \exp(-c_2 \log k \log_2 k) + O(x^{1/2+\varepsilon})$$

$$\ll x\exp(-c_3 \log_3 x \log_4 x).$$

Combining (3.1), (3.2) and (3.3) it follows that

$$\sum_{n \leq x, \omega(n) > a(n)} 1 = [x] - S(x) =$$

$$x + O(x\exp(-C\log_3 x \log_4 x)),$$

as asserted.

For the proof of Theorem 2 we may consider $r > 0$ only, since for $n_k = (p_1 p_2 \ldots p_k)^2$ we have

$$\lim_{k\to\infty} \omega(n_k)/a(n_k) = \lim_{k\to\infty} k2^{-k} = 0.$$

Suppose then that $r > 0$ and $0 < \varepsilon < r/2$ are given. Using (2.4) it is seen that we may find an integer $u > 1$ such that

$$(r - \varepsilon)P^m(u) \geq m \tag{3.4}$$

for $m \geq m_0$. Further for $m \geq m_0$ there exists an integer $k = k(m,r,\varepsilon)$ such that

$$k - 1 < (r - \varepsilon)P^m(u) \leq k,$$

and in view of (3.4) $k \geq m$. We must have $k \leq rP^m(u)$, since otherwise

$$rP^m(u) < k < 1 + (r - \varepsilon)P^m(u),$$

implying $\varepsilon P^m(u) < 1$, which is impossible for m large enough. Therefore

$$(r - \varepsilon)P^m(u) \leq k \leq rP^m(u), \tag{3.5}$$

and taking $n_m = (p_1 p_2 \ldots p_m)^u p_{m+1} \ldots p_{k-1} p_k$ we have

$$r - \varepsilon \le kP^{-m}(u) = \omega(n_m)/a(n_m) \le r ,$$

which proves Theorem 2, since ε may be arbitrarily small and we make $m \to \infty$.

The idea for a result like (2.5) originates with P. Turán [10], who proved

$$\sum_{n\le x} (\omega(n) - \log\log n)^2 << x\log\log x \tag{3.6}$$

thus providing a simple proof of the classical result of Hardy & Ramanujan ([9], pp. 262-275) that almost all integers have about $\log\log n$ distinct prime factors. Therefore our asymptotic formula (2.5) may be considered as a "weighted" analogue of (3.6). Squaring out the expression on the left of (2.5) it is seen that the proof will follow from

$$\sum_{n\le x} \left(\frac{\log\log n}{a(n)}\right)^2 = Bx(\log\log x)^2 + C_1 x\log\log x + O(x) \tag{3.7}$$

$$\sum_{n\le x} \frac{\omega(n)\log\log n}{a^2(n)} = Bx(\log\log x)^2 + C_2 x\log\log x + O(x) \tag{3.8}$$

$$\sum_{n\le x} \frac{\omega^2(n)}{a^2(n)} = Bx(\log\log x)^2 + C_3 x\log\log x + O(x) \tag{3.9}$$

where C_1, C_2, C_3 are suitable constants and

$$B = \prod_p \{1 + \sum_{j=2}^{\infty} (P^{-2}(j) - P^{-2}(j-1))p^{-j}\}. \tag{3.10}$$

To prove (3.7)-(3.10) we proceed similarly as in the proof of (9.9) in [1].

Note that $a^{-2}(n)z^{\omega(n)}$ is a multiplicative function of n, so that for $\operatorname{Re} s > 1$ and $|z| \le 1$

$$\sum_{n=1}^{\infty} a^{-2}(n)z^{\omega(n)}n^{-s} = \prod_p (1 + zp^{-s} + 2^{-2}zp^{-2s} + 3^{-2}zp^{-3s} + 5^{-2}zp^{-4s} + \dots)$$

$$= \zeta^z(s)G(s,z),$$

where

$$G(s,z) = \prod_p (1 - p^{-s})^z (1 + zp^{-s} + 2^{-2}zp^{-2s} + 3^{-2}zp^{-3s} + 5^{-2}zp^{-4s} + \ldots)$$

is absolutely and uniformly convergent for $\mathrm{Re}\, s > 1/2$ and $|z| \le C$ for any fixed $C > 0$. Using a well-known convolution result of A. Selberg ([1], Lemma 2.1) it follows that

$$(3.11) \qquad \sum_{n \le x} a^{-2}(n) z^{\omega(n)} = \frac{G(1,z)}{\Gamma(z)} x \log^{z-1} + R(x,z) ,$$

where uniformly for $|z| \le 3/2$ we have $R(x,z) <<$ $<< x(\log x)^{\mathrm{Re}\, z-2} << x\log^{-1/2} x$. Differentiating (3.11) with respect to the complex variable z we obtain, for $|z| \le 1$,

$$(3.12) \qquad \sum_{n \le x} a^{-2}(n)\omega(n) z^{\omega(n)-1} = \frac{d}{dz}\left(\frac{G(1,z)}{\Gamma(z)}\right) x\log^{z-1} x +$$

$$+ \frac{G(1,z)}{\Gamma(z)} x\log^{z-1} x \log\log x + O(x\log^{-1/2} x),$$

$$(3.13) \qquad \sum_{n \le x} a^{-2}(n)\omega(n)(\omega(n) - 1) z^{\omega(n)-2} = \frac{d^2}{dz^2}\left(\frac{G(1,z)}{\Gamma(z)}\right) x\log^{z-1} x +$$

$$+ 2\frac{d}{dz}\left(\frac{G(1,z)}{\Gamma(z)}\right) x\log^{z-1} x \log\log x +$$

$$+ \frac{G(1,z)}{\Gamma(z)} x\log^{z-1} x (\log\log x)^2 + O(x\log^{-1/2} x) ,$$

where we used Cauchy's inequality for derivatives of analytic functions to bound $\frac{\partial^k}{\partial z^k} R(x,z)\,(k=1,2)$. Setting $z = 1$ in (3.11)-(3.13), adding (3.12) to (3.13) and observing that $\frac{G(1,1)}{\Gamma(1)} = B$ (as defined by (3.10)), we obtain (3.7)-(3.9) by partial summation or by using $\log\log n = \log\log x + O(1)$ for $\sqrt{x} \le n \le x$. Finally a calculation shows that $A = C_1 - 2C_2 + C_3 >$ > 0, completing the proof of (2.5). In concluding it may be mentioned that the asymptotic formula (2.5) could be further sharpened (by introducing new main terms in place of $O(x)$) by using the methods developed in Ch. 5 of [1].

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Received by the editors January 29, 1984.

REZIME

O BROJU ABELOVIH GRUPA DATOG REDA I
BROJU PROSTIH FAKTORA CELOG BROJA

Neka $a(n)$ i $\omega(n)$ označavaju broj neizomorfnih Abelovih grupa sa n elemenata i broj različitih prostih faktora od n respektivno. Raspodela vrednosti $a(n)$ (koja je multiplikativna) i $\omega(n)$ (koja je aditivna) je uporedjena na nekoliko načina.