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FIXED POINT THEORY IN PROBABILISTIC METRIC SPACES

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ABSTRACT

Most fixed point theorems for Probabilistic Metric spaces (PM-spaces) have been proved for the same subclass of PM-spaces. It is shown that this subclass is metrizable. Furthermore, the compatible metric d is related to the distribution functions by

$$d(x,y) < t \quad \text{if and only if} \quad F_{x,y}(t) > 1-t \ .$$

This allows an exact translation of the contraction condition, as well as other conditions studied in metric spaces, to PM-spaces. Thus, theorems follow immediately from corresponding theorems for metric spaces.

1. INTRODUCTION

A real-valued function defined on the set of real numbers is a distribution function if it is nondecreasing, left continuous and $\inf f = 0$, $\sup f = 1$. H denotes the distribution function defined by $H(x) = 0$ if $x \leq 0$, and $H(x) = 1$ for $x > 0$.

DEFINITION 1.1. *Let* X *be a set and* F *be a function on* $X \times X$ *such that* $F(x,y) = F_{xy}$ *is a distribution function. Consider the following conditions:*

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I. $F_{x,y}(0) = 0$ *for all* x,y *in* X.

II. $F_{x,y} = H$ *if and only if* $x = y$.

III. $F_{x,y} = F_{y,x}$.

IV. *If* $F_{x,y}(\varepsilon) = 1$ *and* $F_{y,z}(\delta) = 1$, *then* $F_{x,z}(\varepsilon+\delta) = 1$.

IV_m. $F_{x,z}(\varepsilon+\delta) \geq T(F_{x,y}(\varepsilon), F_{y,z}(\delta))$.

If F satisfies conditions I and II then it is called a pre-probabilistic metric structure (PPM-structure) on X and the pair (X,F) is called a pre-probabilistic metric space (PPM-space). An F satisfying condition III is said to be symmetric. A symmetric PPM-structure F satisfying IV is a probabilistic metric structure (PM-structure) and the pair (X,F) is a probabilistic metric space (PM-space).

DEFINITION 1.2. *A Menger space is a PM-space that satisfies* IV_m, *where* T *is a 2-place function on the unit square satisfying:*

1. $T(0,0) = 0$, $T(a,1) = a$,
2. $T(a,b) = T(b,a)$,
3. *if* $a \leq c$, $b \leq d$, *then* $T(a,b) \leq T(c,d)$,
4. $T(T(a,b),c) = T(a,T(b,c))$.

T *is called a t-norm.*

Let (X,F) be a PPM-space. For $\varepsilon, \lambda > 0$ and $x \in X$, let $N_x(\varepsilon,\lambda) = \{y : F_{x,y}(\varepsilon) > 1-\lambda\}$.

A T_1 topology $\tau(F)$ on X is obtained as follows: $U \in \tau(F)$ if for each $x \in U$, there exists $\varepsilon > 0$ such that $N_x(\varepsilon,\varepsilon) \subset U$. The study of fixed point theory in probabilistic metric spaces (PM-spaces) was started by Sehgal and Bharucha-Reid [10]. The following definition and theorem appeared in their paper.

DEFINITION 1.3. *A mapping* f *of a PM-space* (X,F) *into itself is a contraction if there exists* k, *with* $0 < k < 1$, *such that for each* $x, y \in X$,

$$F_{fx,fy}(kt) \geq F_{x,y}(t) \text{ for all } t > 0.$$

THEOREM 1.1. *Let* (X,F,T) *be a complete Menger space where* $T(a,b) = \min\{a,b\}$. *If* f *is any contraction, there exists a unique* $p \in X$ *such that* $f(p) = p$. *Moreover,* $\lim f^n(q) = p$ *for each* $q \in X$.

A little thought convinces oneself that this is a reasonable definition in this new setting. Also, if f is a contraction ($d(fx,fy) \leq k\, d(x,y)$) on a complete metric space (X,d), and one makes it into a PM-space in the natural way; that is,

$$F_{x,y}(t) = H(t - d(x,y)),$$

then $F_{fx,fy}(kt) \geq F_{x,y}(t)$. In [1], it was shown that the weaker condition,

$$F_{fx,fy}(kt) \geq F_{x,y}(t) \quad \text{whenever} \quad F_{x,y}(t) > 1-t,$$

is sufficient to obtain the above theorem. As originally given, the theorem required T to be continuous and satisfy $T(x,x) \geq x$. It is easy to see that this forces $T(a,b) = \min\{a,b\}$.

2. BASIC THEOREMS

The following condition is another reasonable generalization of a contraction to PM-spaces.

(c) For $t > 0$, $F_{fx,fy}(kt) > 1-kt$ whenever $F_{x,y}(t) > 1-t$.

REMARK 1. If the metric space (X,d) is made into a PM-space as indicated above; that is, $F_{xy}(t) = H(t-d(x,y))$, then if $d(fx,fy) \leq k\, d(x,y)$, for $0 < k \leq 1$, we have condition (c).

P r o o f. $F_{fx,fy}(kt) = H(kt - d(fx,fy)) \geq H(kt-kd(x,y)) = H(t - d(x,y)) = F_{x,y}(t)$. Now $F_{x,y}(t) = H(t-d(x,y)) > 1-t$ if and only if $F_{x,y}(t) = 1$ if and only if $F_{x,y}(t) > 1-kt$. Condition (c) follows.

We now show that for each PM-space in a class larger than the one described in Theorem 1.1, there exists a compatible metric d such that

$$d(fx,fy) \leq k\, d(x,y) \quad \text{iff (c) holds.}$$

Then, using condition (c) as our definition of a contraction, we have Banach's theorem for PM-spaces as a consequence of Banach's theorem for metric spaces. Actually, a nicer result is obtained that allows you to translate many other fixed point theorems for metric spaces to PM-spaces. The result that makes this possible is:

$$d(x,y) < t \quad \text{iff} \quad F_{x,y}(t) > 1-t \,.$$

THEOREM 2.1. *Let* (X,F) *be a symmetric* PPM-*space such that*

$$F_{x,z}(r+s) \geq \min\{F_{x,y}(r), F_{y,z}(s)\} \,.$$

Let $d(x,y) = \begin{cases} \sup\{\varepsilon : y \notin N_x(\varepsilon,\varepsilon),\ 0 < \varepsilon < 1\}\,, \\ 0 \text{ if } \quad y \in N_x(\varepsilon,\varepsilon) \text{ for all } \quad \varepsilon > 0\,. \end{cases}$

Then

(1) $d(x,y) < t$ *if and only if* $F_{x,y}(t) > 1-t$.

(2) d *is a compatible metric for* $t(F)$.

(3) *If* $f:X \to X$ *and* $0 < k \leq 1$,

(c) *holds if and only if* $d(fx,fy) \leq k\, d(x,y)$.

(4) (X,F) *is complete if and only if* (X,d) *is complete.*

P r o o f. Observe that if $t < r$, $N_x(t,t) \subset N_x(r,r)$. Also, $\bigcap\{N_x(\varepsilon,\varepsilon) : 0 < \varepsilon < 1\} = \{x\}$. For, if $x \neq y$, $F_{x,y} \neq H$. Thus there exists $\varepsilon > 0$ such that $F_{x,y}(\varepsilon) = \delta$ where $0 < \delta < 1$. Set $\delta = 1-\delta_1$ and let $\varepsilon_1 = \min\{\varepsilon, \delta_1\}$. Then $F_{x,y}(\varepsilon_1) \leq F_{x,y}(\varepsilon) = \delta = 1-\delta_1 \leq 1-\varepsilon_1$ gives $y \notin N_x(\varepsilon_1, \varepsilon_1)$.

(1) If $1 < t$, $d(x,y) \leq 1 < t$ and also $F_{x,y}(t) \geq 0 > 1-t$. Suppose $d(x,y) < t \leq 1$. Choose δ such that $d(x,y) < \delta < t \leq 1$. Then $y \in N_x(\delta,\delta)$ and $F_{x,y}(t) \geq F_{x,y}(\delta) > 1-\delta > 1-t$. For, if we assume $y \notin N_x(\delta,\delta)$, then $d(x,y) = \sup\{\ \} \geq \delta$, a contradiction. Conversely, suppose $F_{x,y}(t) > 1-t$ where $0 < t \leq 1$. Then $y \in N_x(t,t)$. If $y \notin N_x(\varepsilon,\varepsilon)$ for all $\varepsilon < t$, $F_{x,y}(t) = \lim_{\varepsilon \to t^-} F_{x,y}(\varepsilon) \leq \lim_{\varepsilon \to t^-} (1-\varepsilon) = 1-t$, a contradiction. Thus there exists $0 < \varepsilon < t$ such that $y \in N_x(\varepsilon,\varepsilon)$. Hence $d(x,y) \leq \varepsilon < t$.

(2) If d satisfies the triangular inequality, it is a metric. Also, (1) shows it is compatible with $t(\mathcal{F})$. We observe that $d(x,y) < \varepsilon_1$ and $d(y,z) < \varepsilon_2$ implies that $d(x,z) < \varepsilon_1 + \varepsilon_2$. For, suppose

$$F_{x,y}(\varepsilon_1) > 1-\varepsilon_1 \quad \text{and} \quad F_{y,z}(\varepsilon_2) > 1-\varepsilon_2 .$$

If $F_{x,y}(\varepsilon_1)$ is the minimum,

$$F_{x,z}(\varepsilon_1+\varepsilon_2) \geq \min\{F_{x,y}(\varepsilon_1),\ F_{y,z}(\varepsilon_2)\} > 1-\varepsilon_1 > 1-(\varepsilon_1+\varepsilon_2)$$

gives $d(x,z) < \varepsilon_1+\varepsilon_2$. The triangular inequality follows.

(3) Suppose $d(fx,fy) \leq k\, d(x,y)$ and $F_{x,y}(t) > 1-t$. Then $d(x,y) < t$ and $d(fx,fy) < kt$. Thus $F_{fx,fy}(kt) > 1-kt$. If (c) holds, let $\varepsilon > 0$ be given. Set $t = d(x,y) + \varepsilon$. $d(x,y) = t-\varepsilon < t$ gives

$$F_{x,y}(t) > 1-t, \text{ and } F_{fx,fy}(kt) > 1-kt$$

follows from (c). Thus $d(fx,fy) < kt = k(d(x,y)+\varepsilon) = kd(x,y)+k\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $d(fx,fy) \leq k\, d(x,y)$.

REMARK 2. Assuming the conditions in Theorem 1.1, we have

$$F_{x,z}(r+s) \geq T(F_{x,y}(r), F_{y,z}(s)) = \min\{F_{x,y}(r), F_{y,z}(s)\} ,$$

the inequality in Theorem 2.1. Also, the inequality in Theorem 2.1 does not require the existence of a t-norm. Condition (c) and the earlier definition of contraction seem to be independent for $0 < k < 1$.

COROLLARY. *Let* (X,F) *be a complete symmetric* PPM-*space such that*

$$F_{x,y}(r+s) \geq \min\{F_{x,y}(r), F_{y,z}(s)\} .$$

Suppose $f: X \to X$ *satisfies (c). Then* f *has a unique fixed point* p. *Also, if* $x \in X$ *and* $x_n = f^n(x)$, *then*

(1) $p = \lim x_n$, *and*

(2) *for* $t \geq \frac{k^{n-1}}{1-k} d(x,fx) = \alpha_n$,

$$1 - F_{x_n,p}(t) \leq \frac{k^{n-1}}{1-k} d(x,f(x)) .$$

P r o o f. The theorem gives a compatible metric d such that $d(fx,fy) \leq k\, d(x,y)$. From Banach's fixed point theorem, f has a unique fixed point p satisfying (1). Also,

$$d(x_n,p) \leq \frac{k^n}{1-k} d(x,fx) < \frac{k^{n-1}}{1-k} d(x,fx) = \alpha_n .$$

From (1) of the Theorem,

$$F_{x_n,p}(\alpha_n) > 1 - \alpha_n .$$

For $t \geq \alpha_n$,

$$F_{x_n,p}(t) \geq F_{x_n,p}(\alpha_n) > 1 - \alpha_n .$$

REMARK 3. Note that the error bound is usable. Given $\varepsilon > 0$, choose $0 < \varepsilon_0 < 1$ and x such that $d(x,fx) < \varepsilon_0$; that is, $F_{x,fx}(\varepsilon_0) > 1 - \varepsilon_0$. For $t \geq \beta = \frac{\varepsilon_0}{1-k} > \alpha_n$,

$$1 - F_{x_n,p}(t) \leq \frac{k^{n-1}}{1-k} d(x,fx) < \frac{k^{n-1}}{1-k} \varepsilon_0 .$$

If $\frac{k^{N-1}}{1-k}\varepsilon_0 < \varepsilon$, then $1 - F_{x_n,p}(t) < \varepsilon$ for all $n \geq N$ all $t \geq \beta$.

We next consider how to translate other contractive type conditions for metric spaces to PM-spaces.

LEMMA. *Let* (X,F) *and* d *be as in Theorem 2.1, and* $0 < k \leq 1$. *Let* $R = R(x,y)$ *be a function such that* $d(x,y) \leq R$.

(C*) $F_{fx,fy}(kt) > 1-kt$ *whenever* $F_{x,y}(t) > 1-t$ *and* $t > R$.

Then (C*) *holds if and only if* $d(fx,fy) \leq kR$.

P r o o f. The proof given for (3) of Theorem 2.1 will work here.

The numbering of the various contractive type conditions are those of Rhoades [9]. Conditions (1),(2) and (3) of [9] have obvious translations using Theorem 2.1. The Lemma can be used on other conditions. We illustrate this with the condition

(24): For $0 \leq k < 1$,

$$d(fx,fy) \leq k \max\{d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)\}.$$

The translation is (C*) of the lemma with $R = \mathrm{Max}\{---\}$. There is a difficulty with this translation since (C*) involves $R = R(x,y)$. Another approach is possible. We translate a condition that gives a common generalization of many of the conditions in [9]. The following theorem was proved by Hicks and Rhoades in [4].

THEOREM 2.2. *Let* (X,d) *be a complete metric space and* $0 \leq k < 1$. *Suppose* f *is a self map of* X, *and there exists an* x *such that*

(A) $d(fy,f^2y) \leq k\, d(y,fy)$

for every $y \in O(x,\infty) = \{x, f(x), f^2(x), \dots\}$. *Then:*

(i) $\lim f^n x = q$ *exists.*

(ii) $d(f^n x,q) \leq \frac{k^n}{1-k}\, d(x,fx)$.

(iii) *If* f *is continuous at* q, $fq = q$

It was pointed out in [9], that conditions (1),(4),(5),(7),(9),(11),(18) and (19) each imply (21) and (21) is equivalent to (21′).

(21′) For $0 \le k < 1$,

$$d(fx,fy) \le k \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}.$$

It was noted in [4], that (21′) implies (A) for all $y \in X$, and for $0 \le k \le \frac{1}{2}$, (24) implies (A). The following general theorem follows from Theorem 2.1 and 2.2.

THEOREM 2.3. *Let* (X,F) *be as in Theorem 2.1 and* f *a self map of* X. *Suppose there exists an* x *such that*

(A′) *for* $t > 0$, $F_{fy,f^2y}(kt) > 1-kt$ *whenever*

$$F_{x,fy}(t) > 1-t \text{ and } y \in O(x,\infty) .$$

Then:

(i) $\lim f^n x = q$ *exists.*

(ii) *If* f *is continuous at* q, $fq = q$.

(iii) *For* $t \ge \frac{k^{n-1}}{1-k} f(x,Tx)$, *we have*

$$1 - F_{x_n,p}(t) \le \frac{k^{n-1}}{1-k} d(x,fx) .$$

Thus, for condinuous f, (A′) is more general than the translation of (21′). Also, (A′) refers only to the distribution function. The compatible metric d satisfying $d(x,y) < t$ if and only if $F_{x,y}(t) > 1-t$ allows the translation of many other concepts and theorems from metric spaces to PM-spaces. The following will serve as an illustration.

Let (X,d) be a metric space and let $\varepsilon > 0$. X is ε-chainable if for every $x,y \in X$, there exists $x_0,x_1,\dots,x_n$ in X such that

$$d(x_i,x_{i+1}) < \varepsilon, \quad i=0,1,\dots,n-1.$$

For PM-spaces the condition becomes

$$F_{x_i,x_{i+1}}(\varepsilon) > 1-\varepsilon, \quad i=0,1,\dots,n-1.$$

A mapping f is called an (ε,λ)-local contraction if

$$d(fx,fy) \leq \lambda d(x,y) \text{ whenever } d(x,y) < \varepsilon.$$

This becomes

$$F_{fx,fy}(\lambda t) > 1-\lambda t \text{ whenever } F_{x,y}(\varepsilon) > 1-\varepsilon \quad \text{and}$$
$$F_{x,y}(\lambda) > 1-\lambda; \quad \text{that is, whenever}$$
$$F_{x,y}(\alpha) > 1-\alpha \quad \text{where } \alpha = \min\{\varepsilon,\lambda\}.$$

Edelstein's Theorem [2] for PM-spaces follows.

THEOREM 2.4. *Let* (X,F) *be a complete* ε*-chainable symmetric PPM-space such that*

$$F_{x,y}(r+s) \geq \min\{F_{x,y}(r), F_{y,z}(s)\}.$$

Suppose $f:X \to X$ is an (ε,λ)-contraction, where $0 < \lambda < 1$. Then f has a unique fixed point p and $\lim_n f^n x = p$ for any x in X.

PROBLEM. Can the condition $F_{x,z}(r+s) \geq \min\{F_{x,y}(r), F_{y,z}(s)\}$ in Theorem 2.1 be replaced by some other reasonable (weaker) conditions ?

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REZIME

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Dokazane su teoreme o nepokretnoj tački za neke klase verovatnosnih metričkih prostora.