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ON THE PROBABILISTIC INNER MEASURE OF NONCOMPACTNESS

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ABSTRACT

In this paper some properties of the probabilistic inner measure of noncompactness are investigated and a fixed point theorem is proved.

Beginning with Bocsan's work [1], remarkable attention has been paid to probabilistic measures of noncompactness (briefly, probabilistic measures) and their applications to fixed point theory [2-7]. Usually probabilistic measure is assumed to have the properties:

1) $\phi_A(t) = 1$ $(\forall t > 0)$ if and only if A is precompact,

2) $\phi_{\overline{co}A} = \phi_A$,

3) $\phi_{A\cup B} = \min\{\phi_A, \phi_B\}$.

Having been suggested by [8], here we show that for getting fixed point theorems it suffices to assume 1) and that

2') $\phi_{\overline{co}\,A} \geq \phi_A$,

3') $\phi_{A\cup\{x\}} \geq \phi_A$ for each singleton $\{x\}$.

Then, as an example, we give the definition of probabilistic inner measure and establish some of its properties and its relation with the inner measure studied in [8-9].

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1. Let us first recall some definitions. In the sequel we shall use the following notations. R (R^+) stands for the set of all real (non-negative) numbers, 2^X - the family of all nonempty subsets of X, B(X)-the family of all bounded subsets of a locally convex space X, $\overline{co}$ A-the closed convex hull of A.

A function $F:R \to [0,1]$ is called a distribution if it is non-decreasing, left-continuous, $\inf F = 0$, $\sup F = 1$. A random normed space is a pair $(X,\mathcal{F})$ of a given linear space X and a family $\mathcal{F}$ of distributions $\{F_x : x \in X\}$ satisfying

a) $F_x(t) = 1$ $(\forall t > 0)$ if and only if $x = \theta$,

b) $F_x(0) = 0$,

c) $F_{cx}(t) = F_x(\frac{t}{|c|})$, $\forall c \neq 0$,

d) $F_{x+y}(t+s) \geq \min\{F_x(t) F_y(s)\}$.

Putting $p_\lambda(x) = \sup\{t : F_x(t) \leq 1-\lambda\}$, $(\lambda \in [0,1])$, we get a seminorm and (X,p_λ) becomes a Hausdorff locally convex space. In what follows by all the topological notions in $(X,\mathcal{F})$ we mean the corresponding ones in (X,p_λ). Let $\{\phi_A : A \in B(X)\}$ be a family of distributions satisfying 1), 2'), 3').

DEFINITION 1. *A mapping* $T:X \to 2^X$ *is said to be probabilistic* ϕ*-condensing if* $\phi_{TA} > \phi_A$ *for every* $A \in B(X)$ *which is not precompact.*

Using the method of Reich in [10], we can prove.

THEOREM 2. *Let* $(X,\mathcal{F})$ *be a quasi-complete random normed space ,* C *a nonempty closed convex subset of* $X, T:C \to 2^C$ *an upper semicontinuous probabilistic* ϕ*-condensing mapping having a bounded range. If* $T(x) = \overline{co}T(x)$, *for every* x *in* C *then there exists* $x_o \in C$ *such that* $x_o \in Tx_o$.

P r o o f. Fixing $z \in C$ we denote $\Phi = \{Y \subset C : z \in Y$, Y is closed, convex and $T(Y) \subset Y\}$. Then $\Phi \neq \emptyset$ (since $C \in \Phi$) and each chain in $(\Phi, \subseteq)$ has a lower bound. So by the Zorn lemma, Φ has a minimal element Z. Denote $V = \overline{co}(T(Z) \cup \{z\})$. Obviously, $V \in \Phi$ and $V \subset Z$, hence $V = Z$. But it follows that Z is bounded and $\phi_{TZ} \leq \phi_Z$, so Z is precompact. Since X is quasi-complete and Z is closed, it must be compact. Being an u.s.c. mapping acting in a compact convex subset Z of a Hausdorff locally convex space X, T has a fixed point by the well-known Ky Fan fixed point theorem.

2. Of course each probabilistic measure with properties 1),2),3) (in particular, the measures α_A and β_A in [1,2]) has properties 1), 2´), 3´). We now present a nontrivial example of probabilistic measure with these properties. Denote $h_{AB}(t) = \sup_{s<t} \inf_{x \in A} \sup_{y \in B} F_{xy}(s)$ and call it the probabilistic nonsymmetric Hausdorff distance between A and B in $B(X)$. Now the <u>probabilistic inner measure</u> of A is defined by $b_A(t) = \sup\{\rho > 0$: there is a finite set $A_f \subset A$ with $h_{AA_f}(t) \geq \rho\}$ for $A \in B(X)$, $t \in R$. Remember that in [3] we defined $\beta_A(t) = \sup\{\rho > 0$: there is a finite set $A_f \subset X$ with $h_{AA_f}(t) \geq \rho\}$, and showed that it coincides with the probabilistic Hausdorff measure introduced by Constantin and Bocsan in [2], where h is replaced by H - the probabilistic Hausdorff distance. Obviously, we have

(1) $$b_A \leq \beta_A$$

It is not difficult to see that b_A is a distribution. Besides, by Proposition 5(8) in [3] (where in the proof A_f was taken in A) we have

(2) $$b_A \geq \alpha_A .$$

From (1) and (2) it follows that b_A has property 1). Further, observe that in the definition of b_A we may replace a finite set by a precompact one, so modifying the proof of Proposition 5(6) in [3] we get property 2´). Property 3´) is also easy to verify. Obviously, in general b_A is not monotone with respect to A so it need not satisfy 2) and 3).

Moreover, modifying the proof of Proposition 5 in [3] and using condition c) of F_X above we easily get the following further properties of b_A:

4) $b_{cA}(t) = b_A\left(\frac{t}{|c|}\right), \ \forall c \neq 0,$

5) $b_{x+A} = b_A$,

6) $b_{A\cup B} \geq \min\{b_A, b_B\}$,

7) $b_{A+B}(t+s) \geq \min\{b_A(t), b_B(s)\}$.

Also, modifying the proof of Proposition 7 in [3] we can see that every probabilistic contraction is probabilistic b-condensing.

3. DEFINITION 3. *A distribution* f *is said to be strict if it is strictly monotone, i.e. for each* $c \in (0,1)$ *the equation* $f(t) = c$ *has at most one solution. Geometrically, it means that the graph of* f *does not contain any horizontal interval outside two lines* $y \equiv 0$ *and* $y \equiv 0$.

In [9] Danes introduced the inner Hausdorff measure as follows:

(3) $\chi(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{-net in } A\}$.

We now modify this notion for a locally convex space (X, p_λ) by putting

$$\chi_\lambda(A) = \inf\{\varepsilon > 0: \text{ there are } x_1,\ldots,x_n \in A \text{ such that } A \subset \cup B_\lambda(x_i,\varepsilon)\}$$

where $B_\lambda(x_i,\varepsilon) = \{x \in X : p_\lambda(x-x_i) < \varepsilon\}$. Obviously this measure has the following properties:

i) $\chi_\lambda(A) = 0$ $(\forall\lambda \in (0,1))$ if and only if A is precompact,

ii) $\chi_\lambda(\overline{co}A) \leq \chi_\lambda(A)$,

iii) $\chi_\lambda(A \cup \{x\}) \leq \chi_\lambda(A)$ for each x in X.

The following result establishes the relation between b_A and χ_λ.

THEOREM 4. *Let* (X,F) *be a random normed space,* b_A *the probabilistic inner measure in* X. *Put*

$$\beta_\lambda(A) = \sup\{t : b_A(t) \le 1-\lambda\} .$$

Then $\chi_\lambda \le \beta_\lambda$. *If* b_A *is strict, we have* $\chi_\lambda = \beta_\lambda$.

Conversely, if χ_λ *is the inner measure which is left-continuous and non-increasing in* λ, *then*

$$(4) \qquad \beta_A(t) = 1 - \sup\{\lambda \in (0,1) : \chi_\lambda(A) \ge t\}$$

is a distribution with properties 1), 2'), 3') *and* $\beta_A \ge b_A$. *Moreover:* b_A *is strict* $\Rightarrow b_A = \beta_A$.

P r o o f. Fixing A and λ we denote $K = \{t : b_A(t) \le 1-\lambda\}$, so $a = \beta_\lambda(A) = \sup K$. First we show that $a \ge \chi_\lambda(A)$. Let $t_o > a$, then $b_A(t_o) > 1-\lambda$. By the definition of b_A we get

$$\sup\{\rho > 0 : \text{there are } x_1,\dots,x_n \in A \text{ with } \sup_{s<t_o} \inf_{x\in A} \max_i F_{xx_i}(s) \ge \rho\} > 1-\lambda .$$

So there are $x_1,\dots,x_n \in A$ such that

$$\sup_{s<t_o} \inf_{x\in A} \max_i F_{xx_i}(s) > 1-\lambda .$$

This implies that there exists an $s_o < t_o$ such that for each $x \in A$ there is an i with $F_{xx_i}(s_o) > 1-\lambda$. This inequality is equivalent to $p_\lambda(x-x_i) < s_o$ (see, for example, [11]). But this implies immediately that $\chi_\lambda(A) \le s_o < t_o$, from this $\chi_\lambda(A) \le$ $\le a = \beta_\lambda(A)$.

Assume now b_A is strict and suppose the contrary that $a > b > c > \chi_\lambda(A)$. Then by (3) there are $x_1,\dots,x_n \in A$ such that for each $x \in A$ there is an i with $p_\lambda(x-x_i) < c$, or equivalently $F_{xx_i}(c) > 1-\lambda$. But it implies

$$h_{A\{x_i\}}(b) = \sup_{s<b} \inf_{x\in A} \max_i F_{xx_i}(s) \ge 1-\lambda .$$

So by the definition of b_A we get $b_A(b) \geq 1-\lambda$. Since b_A is nondecreasing and left-continuous, K is closed, i.e. $a \in K$. But this implies $b_A(a) = b_A(b) = 1-\lambda$, a contradiction to the strictness of b_A and the first part of the theorem is proved.

Now fix A, t and denote $\beta_A(t) = a$. Then we must show that $a \geq b_A(t)$. Suppose the contrary that $a < b_A(t)$. Choose a $\lambda_o \in (0,1)$ so that $0 \leq a \leq b = 1-\lambda_o < b_A(t)$. Then by the definition of b_A, there exist $x_1,\dots,x_n \in X$ such that $\sup_{s<t} \inf_{x\in A} \max_i F_{xx_i}(s) > b$. So there is an $s_o < t$ such that for every $x \in A$ there exists an i with $F_{xx_i}(s_o) > b$ or equivalently, $p_{\lambda_o}(x-x_i) < s_o$. From this $\chi_{\lambda_o}(A) \leq s_o < t$, consequently, $\lambda_o > \sup\{\lambda : \chi_\lambda(A) \geq t\}$, hence $1-\lambda_o = b < \beta_A(t)$, a contradiction. b_A is strict $\Rightarrow b_A = \beta_A$. To prove it, denote $b_\lambda(A) = \sup\{t : b_A(t) \leq 1-\lambda\}$ and recall that $b_A(t) = \sup\{\rho : \exists\{x_i\} \subset A,\ h_{A\{x_i\}}(t) \geq \rho\}$, $\chi_\lambda(A) = \inf\{\varepsilon : \exists\{x_i\} \subset A,\ A \subset \cup B_\lambda(x_i,\varepsilon)\}$, $\beta_A(t) = 1 - \sup\{\lambda : \chi_\lambda(A) \geq t\}$. One can prove that $b_A(t) = 1 - \sup\{\lambda : b_\lambda(A) \geq t\}$, so for proving $b_A = \beta_A$ it suffices to show that $\chi_\lambda = b_\lambda$.

First note that $b_\lambda \geq \chi_\lambda$ without any assumption. Indeed denoting $K = \{t : b_A(t) > 1-\lambda\}$, $a = b_\lambda(A)$ we have $a = \inf K$ (here λ and A being fixed). Take $v \in K$, then $b_A(v) > 1-\lambda$ and hence $\exists\{x_i\} \subset A$, $\exists u < v$ such that $\forall x \in A$ $\exists i$ with $F_{xx_i}(u) > 1-\lambda$ but it implies $A \subset \cup B_\lambda(x_i,u)$ and hence $\chi_\lambda(A) \leq v$. So $\chi_\lambda(A) \leq \inf K = a = b_\lambda(A)$.

Now suppose b_A is strict (i.e. $t < s \Rightarrow b_A(t) < b_A(s)$, except for $b_A(t) = b_A(s) = 0$ or 1). We assume the contrary that $a = b_\lambda(A) > a' > \chi_\lambda(A)$. Then $\exists\{x_i\} \subset A$ such that $\forall x \in A$ $\exists i$ with $p_\lambda(x-x_i) < a'$ but it implies $\inf_{x\in A} \max_i F_{xx_i}(a') \geq 1-\lambda$. So $b_A(t) \geq 1-\lambda$ for each $t > a'$. Since b_A is strict, $\inf K = \inf\{t : b_A(t) \geq 1-\lambda\}$. From this, $a' \geq \inf K = a$, a contradiction.

So $a = b_\lambda(A) = \chi_\lambda(A)$.

REFERENCES

[1] G.Bocşan, *On some fixed point theorems in random normed spaces, Proc.V-th Conference on Probability Theory (1974),153-156. Editura Acad.R.S.Roumania, Bucuresti,1977.*

[2] G.Constantin, G.Bocşan, *On some measure of noncompactness in probabilistic metric spaces, ibid., 163-168.*

[3] D.H.Tan , *On probabilistic condensing mappings, Revue Roum.Math. Pures Appl., 26, 10(1981), 1305-1317.*

[4] D.H.Tan , *A note on probabilistic measures of noncompactness, ibid., 28, 4(1983).*

[5] D.H.Tan ,*Some remarks on probabilistic measures of noncompactness (to appear).*

[6] O.Hadžić, *Fixed point theorems for multivalued mappings in random normed spaces, Zb.rad.Prir.-mat.fak.Novi Sad, 9(1979), 29-36.*

[7] O.Hadžić, M.Stojaković,*Some applications of Bocsan's fixed point theorem, ibid., 10(1980), 37-47.*

[8] A.S.Potapov, B.N.Sadovski, *On a fixed point theorem for condensing mappings, Operator methods in nonlinear analysis, Voronež (1982), 85-88.*

[9] J.Danes, *On the Istraţescu's measure of noncompactness, Bull.Math. Soc.R.S.Roumania, 16(64),1972, 403-406.*

[10] S.Reich, *Fixed point theorems in locally convex spaces, Math. Z. 125(1972), 17-31.*

[11] G.Cain, R.Kasriel, *Fixed and periodic points of local contraction mappings on probabilistic metric spaces,Math. System Theory, 9(1976), 289-297.*

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REZIME

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