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ON THE t-NORMS OF THE HADŽIĆ TYPE AND FIXED POINTS IN PROBABILISTIC METRIC SPACES

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ABSTRACT

It is now well known [4] that the Banach principle for probabilistic contractions is valid in complete Menger spaces under a continuous t-norms whose iterations are equicontinuous at x=1. The aim if this note is to give a chatacterization of this class of t-norms and to show that the above mentioned principle can be obtained from the classical. Thus we obtain an improvement of our similar result in [6] where the Min case was considered.

The terminology and the notations are as in [2,10].

DEFINITION. *We shall say that the continuous* t-*norm* T *is an* h-t-*norm if the family* T_m *defined on* $[0,1]$ *by*

$$T_1(x)=x,\quad T_{m+1}(x)=T(T_m(x),x),$$

is equicontinuous at $x=1$.

Examples of h-t-norms are given in [3,4]. Our following result shows that the h-t-norms have a very simple structure:

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LEMMA 1. *The following statements are equivalent*

A. T *is an* h-t-*norm;*

B. T *is continuous and* $\forall a>0$, $\exists b\geq a$ *such that* $T(b,b)=b<1$.

P r o o f. Suppose that A. holds and let $a>0$ be given. Then there exists $c>0$ such that $T_m(x)>a$, $\forall x\geq c$, $\forall m\geq 1$. Since clearly $\{T_m(c)\}$ is nonincreasing, then it is convergent to some limit $b\geq a$. As

$$T_{2m}(c) = T(T_m(c),\ T_m(c))$$

then $b=T(b,b)$ and we obtain that A. implies B.

Conversely, it is obvious that B. implies A. and the lemma is proved.

REMARK 1. In the proof of A.$\Rightarrow$B. only the left - equicontinuity at 1 of T_m and the right continuity of T_1 is used. Clearly, the continuity plays no role in B: $\Rightarrow$A.

REMARK 2. The h-t-norms were considered by O.Hadžić who also constructed an example different from Min [3,4].

The following lemma shows how to construct generalized metrics on a Menger space under an h-t-norm:

LEMMA 2. *Let* T *be an* h-t-*norm. For* $0<a_1<a_2<,\ldots \ldots<a_n\to\infty$, $0<b_1<b_2<\ldots<b_n\to 1$, $T(b_n,b_n)=b_n$ *let us set*

$$F(x) = \begin{cases} 0 & \text{if } x\leq a_1 \\ b_n & \text{if } x\in(a_n,a_{n+1}],\ n=1,2,\ldots \end{cases}$$

Consider a Menger space (S,F,T) *and define*

$$d(p,q)=\inf\{a>0,\ F_{pq}(ax)\geq F(x),\forall x\in R\}$$

Then (i) d *is a generalized metric on* S;

(ii) *If* S *is* F-*complete then* S *is* d-*complete;*

(iii) *The* d-*topology is not weaker than the* F-*topology.*

P r o o f. (i) We prove only the triangle inequality. If $d(p,q) < a' < a$, $d(q,r) < b' < b$ and $x \in (a_n, a_{n+1}]$ then

$$F_{pr}(a'x+b'x) \geq T(F_{pq}(a'x), F(b'x)) \geq$$

$$\geq T_2(F(x)) = T(b_n, b_n) = b_n = F(x) .$$

Therefore $d(p,q) \leq a'+b' < a+b$, and we obtain the triangle inequality.

(ii) and (iii): Let $\{p_n\}$ be a d-Cauchy sequence and fix $a > 0$. By the definition of d, there exists $n_a \geq 1$ such that

$$F_{p_n p_{n+m}}(ax) \geq F(x), \ \forall n \geq n_a, \forall m \geq 1, \ \forall x \in R.$$

If $\epsilon > 0$ and $\lambda \in (0,1)$ are given, then let $a > 0$ and $z_o \in R$ such that $F(z_o) > 1-\lambda$ and $az_o \leq \epsilon$.

If $n \geq n_a$, $m \geq 1$ then $F_{p_n p_{n+m}}(\epsilon) > 1-\lambda$, which shows that p_n is F-Cauchy. If we suppose that S is F-complete then $\{p_n\}$ is F-convergent to some limit p. Therefore

$$F(x) \leq \lim_{m\to\infty} F_{p_n p_{n+m}}(az) = F_{p_n p}(az),$$

for each real z and all $n \geq n_a$, that is $d(p_n,p) \leq a, \forall n \geq n_o$. Thus $\{p_n\}$ is d-convergent and the lemma is proved.

REMARK. For given p_o, q_o in S we can take a_n in the lemma such that $F_{pq}(a_n) \geq b_n$ and the metric d is nontrivial in this case.

The following result was proved in [4]:

THEOREM A. *If* (S,F,T) *is a complete Menger space under an* h-t-*norm then each probabilistic contraction on* S *has a unique fixed point which is the limit of the succesive approximations.*

REMARK 3. As it is well known [7,8,1,2] the Banach contraction principle is a consequence of the above Theorem A. We will prove the following.

THEOREM B. *The Banach fixed point principle implies Theorem A.*

P r o o f. Let (S,F,T) and f be as in Theorem A. If p_o is given in S′ then let a_n and b_n be as in Lemma 2 and such that $F_{p_o f p_o}(a_n) \geq b_n$. Consider the generalized metric d as in Lemma 2. It is each to see that $d(p_o, fp_o) < \infty$ and therefore [5] $S_o = \{q_o \in S, d(p_o, q_o) < \infty\}$ is a complete <u>metric</u> space and f is a contraction in S_o. Therefore $p_n = f^n p_o$ d-converges to the (evidently unique) fixed point of f and the theorem follows.

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REZIME

O t-NORMAMA HADŽIĆ TIPA I NEPOKRETNE TAČKE
U VEROVATNOSNIM METRIČKIM PROSTORIMA

Dat je nov dokaz rezultata iz [4] i [6] i ispitana struktura h-t-normi.