*Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu*
*knjiga 13 (1983)*
*Review of Research Faculty of Science-University of Novi Sad, Volume 13(1983)*

# ON A CLASS OF FUNCTIONS

*Harry I. Miller*[1)]

*Odsjek za matematiku Prirodno-matematičkog fakulteta*
*Univerziteta u Sarajevu, Vojvode Putnika 43, Jugoslavija*

## ABSTRACT

In [3] H. Steinhaus introduced the concept of a permutation function of the interval $[0,1)$ and proved several theorems about these functions. A. Mookhopadhyaya [2] and H. Miller [1] each have several results dealing with Steinhaus permutation functions.

The purpose of this paper is to consider another class of functions, which we will call "switch functions", and show that they share certain properties with the Steinhaus permutation functions.

1. PRELIMINARIES. Each number $t \in [0,1)$, can be writen in one and only one way in dyadic form. This is not quite true that is some numbers, those having finite dyadic representations, have two representations; for example

$$1/2 = \sum_{n=1}^{\infty} 1/2^{n+1}.$$

In such cases we always assign the finite representation to the number under consideration, and in this sense each number in $[0,1)$ has one and only one representation.

1) *The research for this paper was supported by the Foundation for Scientific Work of the Republic of Bosna i Herzegovina.*

*AMS Mathematics subject classification (1980): 26A30, 28D05*
*Key words and phrases: Steinhaus permutation functions, Lebesgue measure.*

By a permutation P of the natural numbers we shall mean any one-to-one function of the natural numbers onto themselves.

As mentioned above each $t \in [0,1)$ can be written in the form

$$t = \sum_{n=1}^{\infty} (e_n/2^n) ,$$

where each $e_n$ is either 0 or 1.

For simplicity we will write this formula as

$$t = 0 \cdot e_1 e_2 e_3 \ldots .$$

If P is a permutation, then by applying P to the indices on the right hand side of the equation for t we obtain a new development which corresponds to a real number $t'$ in $[0,1)$ given by

$$t' = 0 \cdot e_1' e_2' e_3' \ldots \quad , \text{ where } e_n' = e_{P(n)}$$

for each $n = 1,2,3,\ldots$ .

For convenience we shall write $P(t) = t'$. That is the same symbol, i.e. P, is used for two different functions, but no confusion will occur as it will always be clear which P we are discussing from the context in which it occurs.

Many facts about Steinhaus permutations functions(i.e. functions of the type $P : [0,1) \to [0,1)$, $P(t) = t'$ described above) are known. In the following some of the most important facts about Steinhaus permutation functions will be listed.

1) If E is a Lebesgue measurable subset of $[0,1)$ and P is any Steinhaus permutation function, then

$$P(E) = \{ P(e) : e \in E \}$$

is a Lebesgue measurable set and

$$m(P(E)) = m(E) ,$$

where m denotes Lebesgue measure. Two different proofs of this fact can be found in the literature (H.Steinhaus [3] and H.Miller [1]).

2) Each Steinhaus permutation function is continuous at each point of $[0,1)$ with the exception of at most countabily many points. The proof of this theorem is given in the paper of A.Mookhopadyaya [2].

3) Suppose that P is a Steinhaus permutation function that moves infinitely many natural numbers (i.e. $P(n)\neq n$ for infinitely many n). Then it follows that $P'(x)$, the derivative of P at x, exists nowhere on $[0,1)$. This result is due to H.Miller and can be found in [1].

We will now consider a new class of functions which we will call switch functions.

If $t,y \in [0,1)$ and

$$t = 0\cdot e_1 e_2 e_3 \ldots ,$$

$$y = 0\cdot y_1 y_2 y_3 \ldots$$

are the unique (adopting the earlier mentioned convention) binary developments of t and respectively, then $S_y(t)$ is defined by the formula

$$S_y(t) = 0\cdot e_1' e_2' e_3' \ldots ,$$

where

$$e_n' = e_n \quad \text{if} \quad y_n = 0 \quad \text{and}$$

$$e_n' = \hat{e}_n \quad \text{if} \quad y_n = 1,$$

where $\hat{}$ is the switch operation, i.e. $\hat{0} = 1$ and $\hat{1} = 0$. The function $t \to S_y(t)$ (having domain $[0,1)$) will be called the switch function determined by y.

In this paper switch function analogues of results about Steinhaus permutation functions will be proved.

2. RESULTS. Let $\mathcal{P}$ denote the collection of all Steinhaus permutation functions and let $\mathcal{S}$ denote the collection of all switch functions. In our first result we will show that the only function in $\mathcal{P} \cap \mathcal{S}$ is the identity function defined on $[0,1)$.

THEOREM 1. *$P \cap S = \{i\}$, where i denotes the identity function on $[0,1)$, i.e. $i(x) = x$ for every $x \in [0,1)$*

P r o o f. Suppose that P is a permutation of the natural numbers and $P(i) = j$, with $i \neq j$.

Then $(P(x))_i$, the $i^{th}$ number in the expansion of $P(x)$, is given by

$$(P(x))_i = x_j \ , \quad \text{where} \quad x = 0 \cdot x_1 x_2 x_2 \ \dots \ .$$

Furthermore $(S_y(x))_i$, the $i^{th}$ number in the expansion of $S_y(y)$, is given by

$$(S_y(x))_i = x_i \quad \text{or} \quad \hat{x}_i \ ,$$

depending on whether $y_i$ is 0 or 1, where $y = 0 \cdot y_1 y_2 y_3, \dots$ .

In any case by the statistical independence of the numbers appearing in the $i^{th}$ and $j^{th}$ places of the binary developments of numbers in $[0,1)$ we have

$$m(x \in [0,1) : (P(x))_i \neq (S_y(x))_i) = 1/2 \ .$$

From this it follows that $P \neq S_y$ for every $y \in [0,1)$.

We next prove the analogue of 2) in section 1.

THEOREM 2. *If $S \in S$, then S is continuous on $[0,1)$ with the exception of at most countabily many points.*

P r o o f. If $S \in S$, then $S = S_y$ for some $y \in [0,1)$. Consider the sequence of functions $(S_y^n)_{n=1}^{n=\infty}$ defined as follows. For each $x = 0 \cdot x_1 x_2 x_3 \ \dots$ and each natural number n

$$((S_y^n)(x))_i = (S_y(x))_i \quad \text{for} \quad i = 1, 2, \dots, n$$

and

$$((S_y^n)(x))_i = x_i \quad \text{for all} \quad i > n.$$

Then the sequence $(S_y^n)_{n=1}^{n=\infty}$ has the following properties.

a) $(S_y^n)'(x)$ , the derivative of $(S_y^n)$ at x, equals one for each $x \in [0,1)\setminus C_n$, where $C_n$ is a finite set for each n.

b) The sequence $(S_y^n)_{n=1}^{n=\infty}$ converges uniformly to $S_y$ on $[0,1)$.

From a) and b) it is immediate that $S_y$ is continuous at each point of the set $[0,1)\setminus \bigcup_{n=1}^{\infty} C_n$.

The next result is an analogue of 1) in section 1.

THEOREM 3. *If E is a Lebesgue measurable subset of* $[0,1)$ *and* $S_y$ *is any switch function, then*

$$S_y(E) = \{S_y(e) : e \in E\}$$

*is a Lebesgue measurable set and*

$$m(S_y(E)) = m(E) .$$

P r o o f. Let the sequence $(S_y^n)_{n=1}^{n=\infty}$ be defined as in the proof of Theorem 2. Since $(S_y^n)'(x)$ exists for all x in $[0,1)\setminus C_n$, where $C_n$ is a finite set, it follows that each function $S_y^n$ is Lebesgue measurable (in fact is a Baire function of class one). Therefore, by b) in the proof of Theorem 2, it follows that $S_y$ is Lebesgue measurable (in fact is a Baire function of class two). Let B be any Borel subset of $[0,1)$. Define

$$(S_y)^{-1}(B) = (x \in [0,1) : S_y(x) \in B) .$$

It is not difficult to see that the symmetric difference of the sets

$$S_y(B) \quad \text{and} \quad (S_y)^{-1}(B)$$

is an at most countable set, where the symmetric difference of any two sets M and N is defined to be the set $(M\setminus N) \cup (N\setminus M)$. Therefore $S_y(B)$ is a Lebesgue measurable set for each Borel set B (in fact $S_y(B)$ is a Borel set).

The remainder of the proof follows the proof of Theorem 2 in [1], but is included for completeness.

By Theorem 2, $S_y$ is continuous on a set $[0,1)\setminus C$, where C is at most countable. Let $B'$ denote the set $B\setminus C$. Clearly $m(B')=m(B)$. Furthermore, for every $\varepsilon>0$, there exists $C_\varepsilon$, a closed subset of $B'$, such that

$$m(B') - m(C_\varepsilon) < \varepsilon .$$

We will now show that for each $x \in [0,1)$,

$$\limsup_{n\to\infty} X_{S_y^n(C_\varepsilon)}(x) \le X_{S_y(C_\varepsilon)}(x) .$$

Here $X_D$ denotes the characteristic function of the set D, i.e. $X_D(x)=1$ if $x\in D$ and $X_D(x)=0$ if $x\notin D$. To see this suppose that $x\in[0,1)$ and $\limsup_{n\to\infty} X_{S_y^n(C_\varepsilon)}(x)=1$. Then there exists a subsequence $(n_k)_{k=1}^{k=\infty}$ of the positive integers, with $x\in S_y^{n_k}(C_\varepsilon)$ for each integer k. Therefore for each k there exists $e_k\in C_\varepsilon$, such that $x=S_y^{n_k}(e_k)$. There is a subsequence $(e_{k_j})_{j=1}^{j=\infty}$ of the sequence $(e_k)_{k=1}^{k=\infty}$ such that the $\lim_{j\to\infty} e_{k_j}$ exists, that is $\lim_{j\to\infty} e_{k_j}=e$.

However $C_\varepsilon$ is a closed set and therefore $e\in C_\varepsilon$. This in turn implies that

$$S_y(e) = \lim_{j\to\infty} S_y^{n_{k_j}}(e_{k_j}) ,$$

since the sequence $(S_y^n)_{n=1}^{n=\infty}$ converges uniformly to $S_y$ on $[0,1)$, e is a point of continuity of $S_y$ and $\lim_{j\to\infty} e_{k_j}=e$. But $x=S_y^{n_{k_j}}(e_{k_j})$ for each $j=1,2,\dots$ .

Therefore $S_y(e)=x$, with $e\in C_\varepsilon$ and hence

$$\chi_{S_y(C\ )} = 1.$$

Therefore we have shown that for each $x \in [0,1)$,

$$\limsup_{n\to\infty} \chi_{S_y^n(C_\varepsilon)}(x) \le \chi_{S_y(C_\varepsilon)}(x) .$$

This implies that $\lim\limits_{n\to\infty} f_n(x) \le \chi_{S_y(C_\varepsilon)}(x)$ for every $x \in [0,1)$, where

$$f_n(x) = \sup_{k\ge n} \chi_{S_y^k(C_\varepsilon)}(x) .$$

Therefore $\lim\limits_{n\to\infty} (f_n(x) - \chi_{S_y(C_\varepsilon)}(x)) \le 0$ and by the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \le \int_0^1 \chi_{S_y(C_\varepsilon)}(x)\,dx .$$

Furthermore

$$\int_0^1 f_n(x)\,dx \ge \int_0^1 \chi_{S_y^n(C_\varepsilon)}(x)\,dx = m(S_y^n(C_\varepsilon)) .$$

By a) in the proof of Theorem 2 it follows that

$m(S_y^n(C_\varepsilon)) = m(C_\varepsilon)$ and therefore we have

$$m(S_y(C_\varepsilon)) = \int_0^1 \chi_{S_y(C_\varepsilon)}(x)\,dx \ge m(C_\varepsilon) \ , \qquad \forall \varepsilon > 0.$$

From this it follows that

$$m(S_y(B)) \ge m(B)$$

for every Borel set B contained in $[0,1)$.

If $m(S_y(B)) > m(B)$ for some Borel subset of $[0,1)$, then we would have

$$m(S_y(B)) + m(S_y(B^c)) > m(B) + m(B^c) \ ,$$

where $B^c = [0,1) \setminus B$; which implies $1 > 1$.

Therefore we have shown that

$$m(S_y(B)) = m(B)$$

for every Borel subset $B$ of $[0,1)$.

Finally if $E$ is any measurable subset of $[0,1)$ then there exist Borel sets $B_1$ and $B_2$, such that

$$B_1 \subseteq E \subseteq B_2 \subseteq [0,1), \text{ and}$$

$$m(B_1) = m(E) = m(B_2).$$

However, $S_y(B_1) \subseteq S_y(E) \subseteq S_y(B_2)$, and therefore $S_y(E)$ is Lebesgue measurable and

$$m(E) = m(S_y(E)), \text{ concluding the proof.}$$

Our next result is an analogue of 3) in section 1.

THEOREM 4. *If* $y = 0 \cdot y_1 y_2 \ldots \in [0,1)$ *and* $(n : y_n = 0)$ *and* $(n : y_n = 1)$ *are both infinite sets, then* $(S_y)'(x)$ *, the derivative of* $S_y$ *at* $x$*, exists nowhere in* $[0,1)$.

Proof. Let $x \in [0,1)$ and let

$x = 0 \cdot x_1 x_2 x_3 \ldots$ be its binary development.

Define the sequence $(h_n)_{n=1}^{n=\infty}$ in the following way:

$$h_n = 1/2^n \text{ if } x_n = 0 \text{ and } h_n = -1/2^n \text{ if } x_n = 1.$$

A simple calculation shows that:

$$\frac{S_y(x_n + h_n) - S_y(x_n)}{h_n} = \begin{cases} 1 & \text{if } (x_n, y_n) = (0,0) \\ -1 & \text{if } (x_n, y_n) = (0,1) \\ 1 & \text{if } (x_n, y_n) = (1,0) \\ -1 & \text{if } (x_n, y_n) = (1,1) \end{cases}$$

Since the sets $(n:y_n = 0)$ and $(n:y_n = 1)$ are both assumed to be infinite it follows that the sequence

$$\left\{ \frac{S_y(x_n + h_n) - S_y(x_n)}{h_n} \right\}_{n=1}^{n=\infty}$$

contains infinitely many minus ones and infinitely many ones and therefore $(S_y)'(x)$ does not exist.

The next theorem is the switch function analogue of Theorem 3 in [1].

THEOREM 5. *If* $E \subseteq [0,1)$, $m(E) = \gamma > 0$ *and* $\varepsilon > 0$, *then there exists* $y_0$, $0 \le y_0 \le 1$, *such that* $0 \le y \le y_0$ *implies* $m(E \cap S_y(E)) > \gamma - \varepsilon$.

P r o o f. The proof of this theorem will not be given since it is completely analogous to the proof of Theorem 3 in [1].

## R E F E R E N C E S

[1] Miller, H. I.: *On the permutation functions of Steinhaus, Akademija Nauka i Umjetnosti BiH, LXI (1978), 27-35.*

[2] Mookhopadhyaya, A. K.: *On sets of positive measure under certain transformations, Matematički Vesnik, Vol. 1(16), 4, 1964, 303-306.*

[3] Steinhaus, H.: *Sur une transformation des ensembles lineaires, Mathematica, Vol. 3(26), 1, 1961, 179-182.*

Received by the editors September 1, 1983.

**REZIME**

O JEDNOJ KLASI FUNKCIJA

U radu [3] H.Steinhaus je uveo pojam permutacione funkcije intervala [0,1] i dokazao nekoliko teorema o ovim funkcijama. A.Mookhopadhyaya [2] i H.Miller [1] takodje imaju nekoliko rezultata koji se odnose na Steinhausove permutacione funkcije. U ovom radu se definiše klasa "switch funkcija" i dokazuju za ovu klasu funkcija rezultati analogni onim koji su dobijeni za Steinhausove permutacione funkcije.