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A GENERALIZATION OF A DIEUDONNE THEOREM FOR A NONADDITIVE SET FUNCTIONS

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ABSTRACT

In this paper the famous Dieudonné theorem is generalized. If M is a family of triangular set functions defined on the family B of all Borel sets of a locally compact set T with regular variations and M is bounded on every open set, then M is uniformly bounded.

1. INTRODUCTION

As it is well-known, the Nikodym boundedness theorem for measures in general fails for algebras of sets (see Example 5., Diestel, Uhl [2] , p.18). But there are uniform boundedness theorems in which the initial boundedness conditions are on some subfamilies of a given σ-algebra; those subfamilies must not be σ-algebras. A famous theorem of Dieudonné [3] states that for compact metric spaces the pointwise boundedness of a family of Borel regular measures on open sets implies its uniform boundedness on all Borel sets. We shall generalize this Dieudonné theorem on a wider class of set functions. The class of finitely additive regular Borel set functions gives nothing new, because each finitely additive regular Borel set function (also in the case of vector measures) is necessarily countably additive - Kupka [7] .

We take in this paper a wider class of real valued set functions, the so called triangular set functions. We prove a generalized Dieudonné type theorem for this class of set functions. Using some modifications we obtain also a generalization of Dieudonné's theorem for semigroup valued set functions.

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2. TRIANGULAR SET FUNCTIONS

Let T be a locally compact space and S a class of subsets of T such that $\emptyset \in S$.

DEFINITION 1. *(Dinculeanu [4] ,p. 303). A set function* $\mu: S \to R$ *is said to be* *regular* *if for every* $A \in S$ *and every* $\varepsilon > 0$ *there exist a compact set* $K \subset A$ *and an open set* $G \supset A$ *such that for every set* $A' \in S$, $K \subset A' \subset G$, *we have*

$$|\mu(A) - \mu(A')| < \varepsilon$$

DEFINITION 2. *A set function* $\mu: S \to R$ *is said to be* *triangular* *if for every* $A, B \in S$, *such that* $A \cap B = \emptyset$ and $A \cup B \in S$, *we have*

$$\mu(A) - \mu(B) \leq \mu(A \cup B) < \mu(A) + \mu(B).$$

and $\mu(\emptyset) = 0$.

The following theorem is important for further characterization of set functions which are both regular and triangular.

THEOREM 1. *Let S be a ring of subsets of* T. *If a set function* μ $S \to R$ *is regular and superadditive, i. e.*
$\mu(A \cup B) \geq \mu(A) + \mu(B)$ for every $A, B \in S$, $A \cap B = \emptyset$
then it satisfies the following condition

(R) *For every* $A \in S$ *and every number* $\varepsilon > 0$ *there exist a compact set* $K \subset A$ *and an open set* $G \supset A$ *such that for every set* $B \in S$ *With* $B \subset G \setminus K$ *we have*

$$|\mu(B)| < \varepsilon$$

P r o o f. It is enough to adapt the proof of Proposition 1. on page 304 in [4].

COROLLARY 1. *If a set function* $\mu: S \to R$ *(S is a ring),* $\mu(\emptyset) = 0$, *has a regular variation, where the variation* $|\mu|$ *is defined in the usual way, i. e.*

$$|\mu|(E) := \sup_{\pi} \sum_{A \in \pi} |\mu(A)| \qquad (E \in S)$$

and the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of S, then μ *satisfies the condition* (R).

P r o o f. Since $|\mu|$ is superadditive - [4] , p.34, we can apply Theorem 1. on $|\mu|$. Then the inequality $\mu \leq |\mu|$ implies our statement.

It is obvious that a triangular set function μ with a regular variation is itself regular.

DEFINITION 3. *A set function* $\mu : S \to R$ *is said to be* <u>*exhaustive*</u> *whenever given a sequence* (E_n) *of pairwise disjoint member of* S, $\lim_{n\to\infty} \mu(E_n) = 0$.

3. UNIFORM BOUNDEDNESS THEOREM

We shall take from now on for the class S the collection B of all Borel sets of a Hausdorff locally compact topological space T. Now we shall formulate the main theorem.

THEOREM 2. *Let* M *be a family of triangular set functions defined on* B *with regular variations. If the set*

$$\{\mu(O),\ \mu \in M\}$$

is bounded for every open set O, *then*

$$\{\ \mu(B);\ \mu \in M, B \in B\ \}$$

is a bounded set.

REMARK 1. We shall assume in the following proofs that T is a compact Hausdorff space. Namely, we can replace T with an Alexandrov one point ω compactification $T \cup \{\omega\}$, taking $\mu(\omega) = 0$ $(\mu \in M)$

We obtain easily the following

COROLLARY 2. *Let* M *be a family of regular scalar measures defined on* B. *If the set*

$$\{|\mu(O)|;\ \mu \in M\}$$

is bounded for every open set O, *then*

$$\{|\mu(B)|;\ \mu \in M,\ B \in B\}$$

is a bounded set.

P r o o f. Let $\nu(B) := |\mu(B)|$ $(B \in B, \mu \in M)$. It is obvious that the family F of all such set functions ν satisfies the conditions of Theorem 2. (by Proposition 24. from [4], p. 319 $|\nu| = |\mu|$ is also regular). So we apply Theorem 2.

In the proof of Theorem 2 we need two lemmas.

LEMMA 1. *Let* μ *be a triangular set function defined on* $\mathcal{B}$ *with a regular variation. Then* μ *is* σ*-subadditive on each sequence of disjoint open sets* (O_n), i. e.

$$\mu\left(\bigcup_{j=1}^{\infty} O_j\right) \leq \sum_{j=1}^{\infty} \mu(O_j)$$

P r o o f of Lemma 1. First, we shall prove that μ is order continuous on open sets, i. e. for each sequence (U_n) of open sets such that $U_j \supset U_{j+1}$ $(j \in N)$ and $\bigcap_{j=1}^{\infty} U_j = \emptyset$ holds

$$\lim_{j\to\infty} \mu(U_j) = 0.$$

For each $\varepsilon>0$ there exists a sequence of compact sets (K_n) such that $K_j \subset U_j$ and

(1) $$|\mu|(U_j \setminus K_j) < \frac{\varepsilon}{2^j} \quad (j \in N).$$

Then there exists $n_o \in N$ such that $\bigcap_{j=1}^{n} K_j = \emptyset$ for all $n \geq n_o$. Let $n \geq n_o$. Then we have

$$\mu(U_n)=\mu(U_n \setminus \bigcap_{j=1}^{n} K_j)=\mu\left(\bigcup_{j=1}^{n} (U_n \setminus K_j)\right)\leq$$

$$\leq|\mu|\left(\bigcup_{j=1}^{n} (U_n \setminus K_j)\right).$$

Hence, since $|\mu|$ is subadditive (i. e. $|\mu|(A\cup B)\leq |\mu|(A) + |\mu|(B)$ for every papir A,B of not necessarily disjoint sets from $\mathcal{B}$-analogously as in [4] , p. 35-36 and p. 16) and nondecreasing, we obtain by (1)

$$\mu(U_n) \leq \sum_{j=1}^{n} |\mu|(U_j \setminus K_j)< \varepsilon$$

for all $n \geq n_o$. Now, let (O_n) be a sequence of disjoint open sets. Then we have

$$\mu\left(\bigcup_{j=1}^{\infty} O_j\right) \le \sum_{j=1}^{\mu} \mu(O_j) + \mu\left(\bigcup_{j=n+1}^{\infty} O_j\right).$$

Taking $n \to \infty$ we obtain

$$\mu\left(\bigcup_{j=1}^{\infty} O_j\right) \le \sum_{j=1}^{\infty} \mu(O_j).$$

The following lemma is given by C. Swartz in [12] as an extract from the elementary proof of the Antosik-Mikusiński diagonal theorem - [1].

LEMMA 2. *Let X be a Banach space. If* $x_{ij} \in X$ $(i,j \in N)$ *such that* $\lim_{j\to\infty} x_{ij} = 0$ $(i \in N)$, $\lim_{i\to\infty} x_{ij} = 0$ $(j \in N)$ and $\|x_{ii}\| \ge \delta > 0$ $(i \in N)$, *then there exist a sequence* (i_n) *of natural numbers and a sequence* (ε_n) *of positive real numbers such that*

$$\left\| \sum_{k=1}^{n-1} x_{i_n i_k} \right\| = \left(\frac{1}{2} - \varepsilon_n\right) \|x_{i_n i_n}\|; \|x_{i_n i_{+g}}\| < 2^{-2}\varepsilon_n \|x_{i_n i_n}\|$$

(in [12] is δ instead of $\|x_{i_n i_n}\|$).

Proof of Theorem 2. It suffices to prove that every point in T belongs to an open set O so that

(2) $$\sup\{\mu(A) : A \subset O (A \in \mathcal{B}), \mu \in \mathcal{M}\} < \infty.$$

Suppose that this is not true. Then there exists a point $x \in T$ such that (2) does not hold for every open set O such that $x \in O$. We shall prove that there exists a sequence of pairwise disjoint open sets (E_n) and a sequence (μ_n) from $\mathcal{M}$ such that

$$\mu_i(E_i) > i \quad (i \in N).$$

For any open set O such that $x \in O$ there exists a Borel set $B \subset O$ and $\mu_1 \in \mathcal{M}$ such that

(3) $$\mu_1(B) > 4 + 2 \sup_{\mu \in \mathcal{M}} \mu(\{x\}).$$

It is easy to prove that the preceding supremum is finite. Since

μ_1 has a regular variation by Corollary 1 there exists a compact set $K\subset B$ and an open set $O'\subset O$, $B\subset O'$ such that

$$\mu_1(B') < 1$$

for each $B'\subset O'\setminus K$. We have by the subadditivity of μ_1

$$\mu_1(K) + \mu_1(B\setminus K) \geq \mu_1(B).$$

Using the preceding inequality, the inequality

$$\mu_1(B \setminus K) < 1$$

and (3) we obatain

$$\mu_1(K) > 3 + 2 \sup_{\mu\in M} \mu(\{x\}).$$

Let $K_1 = K \cup \{x\}$. Then the last inequality implies (directly for $x\in K$) by the triangularity of μ_1 (for $x\notin K$)

$$\mu_1(K_1) > 3 + \sup_{\mu\in M} \mu(\{x\}).$$

By the regularity of μ_1 there exists an open set U such that $O \supset U \supset K_1$ and

$$\mu_1(B'') < 1 \text{ for every } B''\subset U \setminus K_1.$$

The preceding inequality together with the inequality

$$\mu_1(U) \geq \mu_1(K_1) - \mu_1(U\setminus K_1)$$

implies

(4) $$\mu_1(U) > 2 + \sup_{\mu\in M} \mu(\{x\}).$$

Again by the reqularity of μ_1 there exists an open set W such that $\{x\}\subset W\subset U$ and

(5) $$\mu_1(B''') < 1$$

for every $B'''\subset W\setminus\{x\}$.

Let H be an open set such that $x \in H\subset \bar{H}\subset W$ (where $\bar{H}$ is the closure of the set H). Then we have

$$\mu_1(\bar{H}) \le \sup_{A \subset \bar{H} \setminus \{x\}} \mu_1(A) + \mu_1(\{x\})$$

$$\le \sup_{B \subset W \setminus \{x\}} \mu_1(B) + \mu_1(\{x\}).$$

Hence by (5) we obtain

(6) $$\mu_1(\bar{H}) < 1 + \sup_{\mu \in M} \mu(\{x\}).$$

Let $E_1 = U \setminus \bar{H}$. Then we have $E_1 \subset O$ and $E_1 \cap \bar{H} = \emptyset$

By the inequality

$$\mu_1(E_1) + \mu_1(\bar{H}) \ge \mu_1(U) .$$

(4) and (6) we obtain

$$\mu_1(E_1) > 1.$$

Using the preceding procedure, taking in inequality (3)

" $5 + 2 \sup_{\mu \in M} \mu(\{x\})$ " instead of " $4 + 2 \sup_{\mu \in M} \mu(\{x\})$"

and taking into account the facts that: $x \in H$ and the family M is not bounded on H, we obtain the open sets E_2, H_1 ($H_1 \subset H$) and $\mu_2 \in M$ such that $E_2 \cap H_1 = \emptyset$, $x \in H_1$ and

$$\mu_2(E_2) > 2. \text{ We have } E_1 \cap E_2 = \emptyset.$$

Continuing this procedure we obtain a sequence (μ_i) from M and a sequence (E_i) of pairwise disjoint open sets such that

(7) $$\mu_i(E_i) > i \quad (i \in N).$$

We shall prove that μ_i $(i \in N)$ are exhaustive on a sequence (E_n) of disjoint open sets, i. e.

(8) $$\lim_{j \to \infty} \mu_i(E_j) = 0 \quad (i \in N).$$

Since $\bigcup_{j=1}^{\infty} E_j$ is an open set and $|\mu_i|$ are regular, for $\varepsilon > 0$ by Corollary 1 there exists a compact set $K \subset \bigcup_{j=1}^{\infty} E_j$ such that

$\mu_i(C) < \varepsilon$ for each $i \in N$ and each

$C \subset \bigcup_{j=1}^{\infty} E_j \setminus K'$. Since (E_i) is an open cover of K' so there exists $n_o \in N$ such that $K' \subset \bigcup_{j=1}^{n_o} E_j$.

Then we have for $m \geq n_o + 1$

$$\mu_i(E_m) \leq \sup_{C'} \mu_i(C') \leq \sup_{C} \mu_i(C) < \varepsilon\, (i \in N)$$

where $C' \subset E_m \cup (\bigcup_{j=1}^{n_o} E_j \setminus K')$ and $C \subset \bigcup_{j=1}^{\infty} E_j \setminus K'$.

So we obtain (8).

Let $x_{ij} = \mu_i(E_j)/i$. We have by (8) $\lim_{j\to\infty} x_{ij} = 0\ (i \in N)$.

We obatin by the boundedness assumption of the theorem $\lim_{i\to\infty} x_{ij} = 0$ $(j \in N)$. Applying Lemma 2 on the infinite matrix $|x_{ij}|$ $(i,j \in N)$ we obtain a sequence (i_n) from N and a sequence (ε_n) of positive real numbers such that

$$\sum_{k=1}^{n-1} x_{i_n i_k} = (\tfrac{1}{2} - \varepsilon_n)\, x_{i_n i_n} \tag{9}$$

$$x_{i_n i_{n+q}} < 2^{-q} \varepsilon_n\, x_{i_n i_n} \quad (n \in N). \tag{10}$$

Using the triangularity of μ_{i_n} $(n \in N)$ and Lemma 1 we obtain

$$\mu_{i_n}(\bigcup_{k=1}^{\infty} E_{i_k}) \geq \mu_{i_n}(E_{i_n}) - \sum_{k=1}^{n-1} \mu_{i_n}(E_{i_k}) - \sum_{k=n+1}^{\infty} \mu_{i_n}(E_{i_k})$$

$(n \in N)$. Hence by (9) and (10)

$$i_n^{-1} \mu_{i_n}(\bigcup_{k=1}^{\infty} E_{i_k}) \geq x_{i_n i_n} - \sum_{k=1}^{n-1} x_{i_n i_k} - \sum_{k=n+1}^{\infty} x_{i_n i_k} \geq$$

$$\geq \frac{x_{i_n i_n}}{2} \quad (n \in N), \text{ i. e.}$$

$$\mu_{i_n}(\bigcup_{k=1}^{\infty} E_{i_k}) \geq \frac{\mu_{i_n}(E_{i_n})}{2} \quad (n \in N).$$

Then by (7) we obtain

$$\mu_{i_n}\left(\bigcup_{k=1}^{\infty} E_{i_k}\right) \geq \frac{i_n}{2} \quad \text{for each } n \in N.$$

Since $\bigcup_{k=1}^{\infty} E_{i_k}$ is an open set we obtain a contradiction with the boundedness of (μ_{i_n}) on open sets.

4. FURTHER GENERALIZATIONS

Let X be a commutative semigroup with a neutral element 0. Let $d: X \to [0,+\infty)$ be a pseudometric which satisfies the following condition

$$(d_+) \quad d(x+x_1, y+y_1) \leq d(x,y) + d(x_1,y_1)$$

for all $x, x_1, y, y_1 \in X$.

EXAMPLE. Weber [13] has proved that for every commutative complete uniform semigroup there exists a family of pseudometrics which satisfy (d_+) and which generate its uniformity.

Let X be endowed with a pseudometric d which satisfies (d_+). Now we can extend the definition of the regularity of a set function $\nu: S \to X$ only taking in Definition 1 ν and "$d(\nu(A),\nu(A'))<\varepsilon$" instead of μ and "$|\nu(A)-\nu(A')|<\varepsilon$ " respectively.

The pseudometric d induces a <u>triangular functional</u> - E. Pap [8], [10] in the following way

$$f(x) := d(x,0) \quad (x \in X).$$

The functional f satisfies

$$(F_1) \quad f(x+y) \leq f(x) + f(y) \quad \text{and}$$

$$(F_2) \quad f(x+y) \geq |f(x) - f(y)| \quad \text{for all } x,y \in X.$$

Now we define the variation $|\nu|$ of a set function $\nu: \mathcal{S} \to X$ with $\nu(\emptyset) = 0$ in the following way

$$|\nu|(E) := \sup_{\pi} \sum_{A \in \pi} f(\nu(A)) \quad (E \in \mathcal{S})$$

where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of $\mathcal{S}$. It is easy to see that $|\nu|$ is superadditive.

A set function $\nu: \mathcal{S} \to X$ is said to be a semigroup valued triangular set function if it satisfies

$$\nu(\emptyset) = 0,$$

$$f(\nu(A)) - f(\nu(B)) \leq f(\nu(A \cup B)) \leq f(\nu(A)) + f(\nu(B))$$

for $A, B \in \mathcal{S}$ with $A \cap B = \emptyset$.

Now we have the following generalization of Theorem 2.

THEOREM 3. *Let $\mathcal{F}$ be a family of semigroup valued triangular set functions with regular variations defined on $\mathcal{B}$. If the set*

$$\{ f(\nu(O)); \nu \in \mathcal{F} \}$$

is bounded for every open set O, then

$$\{ f(\nu(B)); \nu \in \mathcal{F}, B \in \mathcal{B} \}$$

is a bounded set.

P r o o f. We take $\mu(B) := f(\nu(B))$ $(B \in \mathcal{B}, \nu \in \mathcal{F})$ and we apply Theorem 2.

REMARK 2. Theorem 3. holds also for a family of N-triangular set functions $\nu: \mathcal{B} \to G$ ($(G, | \ |)$ is a quasinormed group) with a constant $N \in (0, \infty)$ - [6], [4], i. e. such that $\nu(\emptyset) = 0$ and

$$|\nu(A)| - N\,|\nu(B)| \leq |\nu(A \cup B)| \leq |\nu(A)| + N|\nu(B)|$$

for all disjoint $A, B \in \mathcal{B}$.

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REZIME

JEDNO UOPŠTENJE TEOREME DIEUDONNEA NA NEADITIVNE SKUPOVE FUNKCIJE

U radu se kao uopštenje teoreme Dieudonné-a dokazuje teorema o uniformnoj ograničenosti familije, u opštem slučaju, neaditivnih skupovinih funkcija. Klasa izučavanih skupovnih funkcija se sastoji od tkzv. trougaonih skupovnih funkcija. $\mu: S \to R$ (S je familija podskupova lokalno kompaktnog prostora i $\emptyset \in S$ je trougaona skupovna funkcija ako za svako $A, B \in S$, tako da je $A \cap B = \emptyset$ i $A \cup B \in S$, važi

$$\mu(A) - \mu(B) \le \mu(A \cup B) \le \mu(A) + \mu(B) \quad \text{i} \quad \mu(\emptyset) = 0.$$

Neka je M familija trougaonih skupovnih funkcija definisanih na familiji B svih Borelovih podskupova lokalno kompaktnog prostora T sa regularnim varijacijama. Ako je familija M ograničena nad svakim otvorenim skupom, tada je ona i uniformno ograničena (teorema 2). Na kraju se dobijeni rezultat prenosi i na skupovne funkcije sa vrednostima u komutativnoj polugrupi i na N-trougaone skupovne funkcije sa vrednostima u komutativnoj grupi sa kvazi-normom [6], [4].