Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 13 (1983)
Review of Research Faculty of Science-University of Novi Sad, Volume 13(1983)

A SIMPLE PROOF OF A GENERALIZED DIEUDONNÉ THEOREM

Endre Pap

Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića br. 4, Jugoslavija

ABSTRACT

A simple proof is given of a Dieudonné type theorem on uniform boundedness of a family of regular measures defined on the family of all Borel sets of an arbitrary Hausdorff topological space.

A famous theorem of Dieudonné states that for a compact metric space T the pointwise boundedness of a family of Borel regular measures on the family of all open sets of T implies its uniform boundedness on the family of all Borel sets of T. There are several generalizations of this theorem [2], [5], [7], [8]. A funitely additive regular Borel set function reduces to a measure even for a Banach space valued set function, as wes pointed out by J. Kupka [5], the proof is given in [3]. In [7] T is a regular (T_3) space and in [2] T is a locally compact space. In this note we shall give an easy proof of a Dieudonne type theorem for the case when T is an arbitrary

---

AMS Mathematics subject classification (1980): 28C15, 28A10
Key words and phrases: Regular measures, Hausdorff topological space, open sets, Borel sets.

Hausdorff topological space. The proof is quite elementary and short. It is based on two lemmas which are connected with certain properties of numbers.

LEMMA 1. (P. Antosik [1].) *Let* x_{ij} $(i,j \in N)$ *be complex numbers. If* $\lim_{j\to\infty} x_{ij} = 0$ *for* $i = 1,2,\ldots,$ *then there exist an infinite set* I *of positive integers and a subset* J *(finite or infinite) of* I *such that for all* $i \in I$ *we have*

$$\sum_{j\in J} |x_{ij}| < \infty$$

and

$$\Big|\sum_{j\in J} x_{ij}\Big| \geq \frac{1}{2}\,|x_{ii}|.$$

REMARK 1. Originally Lemma 1 was stated in [1] for elements from normed space.

Let T be a Hausdorff topological space and B the collection of all Borel subsets of T. A Borel measure μ on T is regular: if B is a Borel subset of $T, \varepsilon > 0$, then there exists a compact subset $K \subset B$ with $|\mu|(B\setminus K) < \varepsilon$.

LEMMA 2. (J.D.Stein [7], Lemma 1.). *If* μ *is a nonzero regular measure on* T *and* U *is an open subset of* T*, then the following holds:*

There exists an open set V *with* $V \subset U$ *and*

$$|\mu(V)| > |\mu|(U)/7 .$$

The easy proof of this Lemma in [7] is based on an inequality for complex number [6]: if $z_1, z_2, \ldots, z_n$ are complex numbers, there is a subset S of $\{1,2,\ldots,n\}$ such that

$$\Big|\sum_{j\in S} z_j\Big| \geq \frac{1}{6}\sum_{j=1}^{n} |z_j| .$$

REMARK 2. Lemma 2 is stated in [7] for the case when T is a regular (T_3) topological space but the same proof holds also when T is a Hausdorff topological space.

THEOREM. *Let* M *be a family of regular scalar measures defineb on* B. *If the set*

$$\{|\mu(O)| \ |\mu \in M\}$$

is bounded for every open set O, *then*

$$\{|\mu(B)| \ |\mu \in M, B \in B\}$$

is a bounded set.

P r o o f. Firstly, let us suppose that the set of variations of the elements of the family M is unbounded on an open set O. Then for some $M_1 > 0 |\mu(O)| < M_1$ $(\mu \in M)$ and for each $M > 0$ and each $\varepsilon > 0$ there exists $\mu \in M$ such that $|\mu|(O) > 7(M+M_1+3\varepsilon)$. By Lemma 2 there exists an open set $W \subset O$ such that $|\mu(W)| > M+M_1+3\varepsilon$. Since μ is regular there exists a compact subset K_1 of W such that $|\mu(W \setminus K_1)| < \varepsilon$. Hence $|\mu(K_1)| > M+M_1+2\varepsilon$. Again by the regularity of μ there exists a compact subset K_2 of $O \setminus K_1$ such that $|\mu|((O \setminus K_1) \setminus K_2) < \varepsilon$, thus $|\mu(K_2)| > M+\varepsilon$. Since $K_1 \cap K_2 = \emptyset$ and T is a Hausdorff topological space there exist disjoint open sets H and V such that $H \supset K_1$ and $V \supset K_2$ such that $|\mu(V \setminus K_2)| < \varepsilon$ and $|\mu(H \setminus K_1)| < \varepsilon$. Then we obtain

$$|\mu(V)| > M \quad \text{and} \quad |\mu(H)| > M .$$

Let us assume that the theorem is not true, i.e.

$$\nu(T) = \infty , \quad \text{where} \quad \nu(B) = \sup_{\mu \in M} |\mu|(B) \quad (B \in B).$$

By the preceeding for any given arbitrary large positive number there exist $\mu_1 \in M$ and two disjoint open subsets H_1 and V_1 of T, such that $|\mu_1(H_1)|$ and $|\mu_1(V_1)|$ are greater than it. Then there are two possibilities. Either one of $\nu(H_1)$ or $\nu(V_1)$ is infinite (in this case we take $\nu(H_1) = \infty$) or both are finite.

In the first case, we apply the preceding procedure on H_1 (instead of T). Then for any given positive number, we can find a measure $\mu_2 \in M$ and disjoint open sets H_2 and V_2 such that $|\mu_2(H_2)|$ and $|\mu_2(V_2)|$ are greater than it. Thus, subsequently, by repeating the process, supposing always that $\nu(H_i) = \infty$, we can obtain a sequence (μ_i) from M and a sequence of disjoint open sets (V_i) such that

$$(1) \qquad |\mu_i(V_i)| > i \quad (i \in N) .$$

Because of $\lim_{j\to\infty} \mu_n(V_i) = 0$ $(n \in N)$ we can apply Lemma 1 on $\mu_i(V_j)$ $(i,j \in N)$. According to this there exist an infinite set $I \subset N$ and its subset J such that for each $i \in I$

$$|\sum_{j\in J} \mu_i(V_j)| \geq \frac{1}{2} |\mu_i(V_i)| .$$

By the preceding inequalities and (1), we obtain $|\mu_i(V)| > i$ $(i \in I)$ for the open set $V = \bigcup_{j\in J} V_j$, contradictory to the condition of the boundedness of the family M on open sets.

Suppose now that both $\nu(H_1)$ and $\nu(V_1)$ are finite. This implies $\nu((T\backslash K_1)\backslash K_2)) = \infty$. Now we can apply the preceding procedure on the open set $T\backslash(K_1 \cup K_2)$ (instead of T). Thus, sequently, by repeating the process, supposing always that $\nu(H_i)$ and $\nu(V_i)$ are finite (i.e. there exists $p_i \in N$ such $\nu(H_i) < p_i$ and $\nu(V_i) < p_i$) we can obtain a sequence (μ_i) from M and a sequence of open sets (in general not disjoint) (V_i) such that

$$(2) \qquad |\mu_i(V_i)| > i + c_{i-1} + \varepsilon \qquad (i \in N) ,$$

where $c_i = \sum_{k=1}^{i} p_k$.

By
$$\nu\left(\bigcup_{k=1}^{i-1} V_k \setminus V_i\right) \le c_{i-1} \qquad (i \in N) \quad ,$$

$$|\mu_i| \left(\bigcup_{k=i+1}^{\infty} V_k \setminus \bigcup_{k=1}^{i} V_k\right) < \varepsilon \qquad (i \in N)$$

and (2) we obtain

$$|\mu_i(V)| > i \qquad (i \in N)$$

for the open set $V = \bigcup_{i=1}^{\infty} V_i$. Again a contradiction.

Finally, we reduce the general case to the preseding two. Namely, in general, we combine the preceding two procedures, always taking, as the initial set for the next step, that the open set for which ν is infinite.

If there is an infinite set $I \subset N$ such that $\nu(H_i) = \infty$ $(i \in I)$, then, passing to a subsequence we reduce all to the first case. In the opposite case, we reduce all to the second case.

REFERENCES

[1] P. Antosik, *On the Mikusiński Diagonal Theorem, Bull. Acad.Polon.Sci.Ser.Math.Astronom.Phys., 19(1971), 305-310.*

[2] J.K. Brooks, *On a Theorem of Dieudonné, Adv. in Math., 36(1980), 165-168.*

[3] J. Diestel, J.J. Uhl, *Vector measures, Math.Surveys 15, AMS, 1977.*

[4] J. Dieudonné, *Sur la convergence des suites de measures de Radon, An.Acad.Brasil.Ci.23 (1951), 21-38, 277-282.*

[5] J. Kupka, *Uniform boundedness principles for regular Borel vector measures, J.Austral.Math.Soc. (Series A) 19 (1980), 206-218.*

[6] W. Rudin, *Real and Complex Analysis, McGraw-Hill, 1966.*

[7] J.D. Stein, *A uniform-boundedness theorem for measures, Michigan Math.J., 19 (1972), 161-165.*

[8] B.B. Wells,Jr., *Weak compactness of measures, Proc. Amer.Math.Soc., 20 (1969), 124-130.*

Received by the editors February 22, 1984.

REZIME

JEDAN JEDNOSTAVAN DOKAZ UOPŠTENJA
DIEUDONNE-OVE TEOREME

U radu je dat jednostavan elementaran dokaz uopštenja Dieudonne-ove teoreme o uniformnoj ograničenosti familije regularnih mera. Neka je M familija regularnih skalarnih mera definisanih nad famijom $\mathcal{B}$ svih Borelovih podskupova Hausdorffovog prostora T. Ako je skup

$$\{|\mu(O)| \ |\mu \in M\}$$

ograničen za svaki otvoren skup O, tada je

$$\{|\mu(B)| \ |\mu \in M, B \in \mathcal{B}\}$$

ograničen skup.