Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 13 (1983)
Review of Research Faculty of Science-University of Novi Sad, Volume 13(1983)

A NOTE
ON SOME CONVERGENCES ON SEMIGROUPS

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ABSTRACT

In the paper a new notion on a commutative semigroup with a non-trivial subadditive and homogeneous functional - Chauchy sequence condition is introduced and with it a general theorem on convergences is obtained. As consequences of this theorem, some Orlicz-Pettis theorems are obtained.

1. INTRODUCTION

The point in the proofs of many Orlicz-Pettis type theorems ([2], [8], [9]) is: if $\sum_n x_n$ is weak subseries convergent then the sequence (x_n) has a subsequence which is norm convergent to 0. The purpose of this paper is to prove a general theorem - Theorem 3.3, on a commutative semigroup which extracts this connection between convergences. Let us observe that even in the classical case the approach is a new one.

An important tool is a generalization of the Hahn-Banach theorem on commutative semigroups ([3], [4]) which gives us the existence of a nontrivial special additive functional.

AMS Mathematics subject classification (1980): 40A04, 40J05
Key words and phrases: Commutative semigroup, subadditive and homogeneous functional, additive functional, H-Cauchy sequence condition.

2. SUBADDITIVE, HOMOGENEOUS AND ADDITIVE FUNCTIONALS

Let X be a commutative semigroup. A functional $f:X \to R_+$ (R_+ is the set of all nonnegative real numbers) will be called a subadditive functional if it satisfies the following condition

(F_1) $f(x+y) \leq f(x) + f(y)$ for all $x,y \in X$.

REMARK 1. H.Weber [10] has proved that for every commutative complete uniform semigrup there exists a family of pseudometrics d which satisfy $d(x+x_1,y+y_1) \leq d(x,y)+d(x_1,y_1)$ for all $x,x_1,y,y_1 \in X$ and which generate its uniformity. Such a pseudometric induces a subadditive functional f in the following way

$$f(x) := d(x,0) \ (x \in X).$$

We say that a functional $f:X \to R_+$ if homogeneous if

(F_2) $f(nx) = n f(x)$ $(x \in X,\ n \in N)$.

The condition (F_2) is independent from (F_1). For example, let (h_k) be a Hamel basis for a vector space, then for $x = \Sigma a_k h_k \in X$ we define $p(x) = \Sigma\sqrt{|a_k|}$. Obviously $p(.)$ is a quasi-norm, but $p(nx) = \sqrt{n}\, p(x)$ for all $n \in N$ and all $x \in X$.

To each subadditive functional we can correspond a homogeneous functional which is closely connected with the original one in the following way.

PROPOSITION 2.1. *Let f be a subadditive functional on a commutative semigroup X. Then there exists a homogeneous functional F on X such that*

(i) F *is subadditive,*

(ii) $F(x) \leq f(x)$ $(x \in X)$:

P r o o f. We take that

$$F(x) = \inf\{\tfrac{1}{n} f(nx) \mid n \in N\} \quad (x \in X).$$

As an easy consequence of the generalized Hahn-Banach theorem from [3] and [4] (also in [7]) we can obtain the following theorem.

THEOREM 2.2. *Let X be a commutative semigroup and f be a homogeneous finite subadditive functional on X. If* x_o *is an element from X such that* $f(x_o) \neq 0$ *, then there exists an additive functional h on X such that* $h(x_o) = f(x_o)$ *and* $h(x) \leq f(x)$ *for all* $x \in X$.

REMARK 2. Condition $f(x_o) \neq 0$ from the preceding theorem implies $nx_o \neq x_o$ for each $n \in N$, i.e. x_o is not of a finite order.

3. MAIN RESULTS

Let X be a commutative semigroup with a neutral element 0 and with a nontrivial homogeneous subadditive functional f.

The following notion will be crucial in the main theorem 3.3 of this section. Let (y_j) be a sequence of elements from X and H be a family of additive functionals defined on X such that $h(x) \leq f(x)$ $(x \in X)$. Then a subset X_1 of X will be called a $((y_j),H)$ - subsemigroup if: x belongs to X_1 iff $h(u_1 + \ldots + u_k) \to h(x)$ as $k \to \infty$ and all $h \in H$ for some sequence (u_j) such that u_j is either $\lambda_j y_j$ (for $\lambda_j \in N$) or 0 and $\sum_{j=1}^{\infty} |h(u_j)| < \infty$ $(h \in H)$.

X_1 is nonempty. Namely, 0 and all the members and the finite sums of the members of the sequence (y_j) belong to X_1. Since the series $\sum_{j=1}^{\infty} h(u_j)$ $(h \in H)$ are unconditionally convergent it is easy to see that X_1 is really a subsemigroup of X.

We need in the proofs of Theorem 3.3 and Theorem 3.4 the following theorem. We always have finite additive functionals.

THEOREM 3.1. *Let* (h_n) *be a sequence of additive functionals on a commutative semigroup* X. *Let* (x_n) *be a sequence from* X *such that for its every subsequence* (z_n) *there exist a subsequence* (y_n) *of* (z_n) *and an element* y *from* X *such that*

$$h_n(y_1 + \dots + y_k) \to h_n(y)$$

as $k \to \infty$ *for each* $n \in N$.

Then there exist an infinite set $I \subset N$ *and an element* x *from* X *such that for all* $n \in I$

$$\sum_{j \in J} |h_n(x_j)| < \infty \quad \textit{for some } J \subset I,$$

$$|h_n(x)| \geq \frac{1}{2} |h_n(x_n)| .$$

Since $h_n(\cdot)$ are triangular functionals (i.e. subadditive and $|h_n(x+y)| \geq |h_n(x)| - |h_n(y)|$ for $x, y \in X$, $|h_n(0)| = 0$) the proof of Theorem 3.1 is analogous to the proof of the Antosik-Mikusiński Diagonal Theorem [1], [5] and [6] (using in the second part of the proof the assumption on sequence (x_n)).

Let H be a family of finite additive functionals h on X with the property $h(x) \leq f(x)$ $(x \in X)$. We say that X satisfies the H-Cauchy sequence condition if for each sequence (y_j) from X such that

$$\sum_{j=1}^{\infty} |h(y_j)| < \infty \qquad (h \in H)$$

and each sequence (h_n) from H such that it is a Cauchy sequence on (y_j), i.e. for each $\varepsilon > 0$ there exists $n_o \in N$ such that

$|h_n(y_j) - h_m(y_j)| < \varepsilon$ for all $n,m \geq n_o = n_o(j)$, $j \in N$, then (h_n) is a Cauchy sequence on the $((y_j),H)$ - subsemigroup.

It is easy to see that if X is a finite semigroup or H is a finite family, then X satisfies the H-Cauchy sequence condition.

In a specially important case we have the following proposition.

PROPOSITION 3.2. *Each normed space X satisfies the B^* - Cauchy sequence condition (B^* is the unit ball in the dual X^*).*

P r o o f. We shall prove that each sequence (h_n) of continuous linear functionals from B^* which is a Cauchy sequence on each member of the sequence (y_j) is a Cauchy sequence on the whole closed linear subspace $\overline{L((y_j))}$ generated by (y_j).

Let $x \in \overline{L((y_j))}$. Then for each $\varepsilon > 0$ there exist $\lambda_1, \ldots$ $\ldots, \lambda_{k_o}$ such that

$$\| \lambda_1 y_1 + \ldots + \lambda_{k_o} y_{k_o} - x \| < \frac{\varepsilon}{3}$$

Since (h_n) is a Cauchy sequence on (y_n) there exists $n_o \in N$ such that

$$|h_m(y_j) - h_n(y_j)| < \frac{\varepsilon}{3|\lambda_j|k_o} \qquad (\lambda_j \neq 0)$$

for each $n,m \geq n_o$ and each $j=1,\ldots,k_o$. Hence we have

$$|h_m(x)-h_n(x)| \leq 2\| \lambda_1 y_1 + \ldots + \lambda_{k_o} y_{k_o} - x\| + \\ + \sum_{j=1}^{k_o} |\lambda_j| |h_m(y_j)-h_n(y_j)| < \varepsilon$$

for each $n,m \geq n_o = n_o(\varepsilon, x)$.

We obtain as a consequence of the Hahn-Banach theorem on normed spaces (i.e. if $s_n \in L((y_j))$ $(n \in N)$ and $h_n(s_n) \to h(s)$ as $n \to \infty$ for each $h \in B^*$, then $s \in \overline{L((y_j))}$)

$$((y_j), B^*) \subset \overline{L((y_j))} \quad .$$

We say that a family H of additive functionals on X with the property $h(x) \leq f(x)$ $(x \in X,\ h \in H)$ satisfies the $\underline{\varepsilon\text{-condition}}$ if for arbitrary $x_o \in X$, each $\varepsilon > 0$ and each additive functional h' on X with the property $h'(x) \leq f(x)$ $(x \in X)$ there exists $h \in H$ such that $h(x_o)+\varepsilon > h'(x_o)$.

If H is the family of all additive functionals with the property $h(x) \leq f(x)$ $(x \in X,\ h \in H)$ then it satisfies trivially the ε-condition.

E.Thomas has introduced in Theorem II.3 from [9] a subfamily H of the dual X^* of a normed space such that $\|x\| = \sup_{x^* \in H \cap B^*} |\langle x, x^* \rangle|$ $(x \in X)$. It is easy to see that such a family satisfies the ε-condition.

Now we have the main theorem.

THEOREM 3.3. *Let X be a commutative semigroup with a neutral element 0 and with a nontrivial finite homogeneous subadditive functional f. Let H be a family of additive functionals on X which satisfies the ε-condition. If X satisfies the H-Cauchy sequence condition and (x_n) is a sequence from X such that for every subsequence (y_n) of (x_n) there exists an element $y \in X$ such that*

$$h(y_1 + \ldots + y_n) \to h(y) \quad as \quad n \to \infty$$

for each $h \in H$, then $f(x_n) \to 0$ as $n \to \infty$.

P r o o f. of Theorem 3.3.

Suppose that the theorem is not true. Then for every $\varepsilon > 0$ there exists a subsequence (z_n) of (x_n) such that $f(z_n) > 4\varepsilon$ $(n \in N)$. By Theorem 2.2 there exists a sequence (h'_n) of additive functionals on X such that

$$h'_n(x) \leq f(x) \quad (x \in X,\ n \in N) \text{ and } h'_n(z_n) > 4\varepsilon .$$

Since H satisfies the ε-condition we have a sequence (h_n) from H such that $h_n(z_n) > 3\varepsilon$ $(n \in N)$. We have

$$h_n(y_1 + \ldots + y_k) \to h_n(y)$$

as $k \to \infty$ $(n \in N)$ for a subsequence (y_n) of (z_n) and $y \in X$. Hence $h_n(y_j) \to 0$ as $j \to \infty$ for each $n \in N$. Then there exists a sequence (j_s) of natural numbers such that

$$(1) \qquad |h_{j_s}(y_{j_{s+q}})| < 2^{-1-q'-q} \qquad (s, q \in N),$$

where q' is a fixed natural number such that $2^{-q'} < \varepsilon$.

Now by the diagonal procedure we shall construct a subsequence of (h_{j_s}) which we denote with (q_n), such the sequence $(g_n(y_{j_s}))$ is convergent for each fix $s \in N$.

Since (x_n) is a sequence from X such that for every subsequence (u_n) of (x_n) there exists an element $u \in X$ such that

$$\sum_{n=1}^{\infty} h(u_n) = h(u) \quad \text{for each } h \in H$$

so we obtain by Riemann's theorem on convergences of series of real numbers

$$\sum_{s=1}^{\infty} |h(y_{j_s})| < \infty \quad (h \in H).$$

X satisfies the H-Cauchy sequence condition so (g_n) is a Cauchy sequence on the $((y_{j_s}),H)$ -semigroup X_1.

Now we take $g_{j_{k+1}} - g_{j_k}$ $(k \in N)$. Then by Theorem 3.1 there exist $x \in X_1$ and an infinite set $I \subset N$ such that

$$|g_{j_{k+1}}(x) - g_{j_k}(x)| \geq \frac{1}{2} |g_{j_{k+1}}(y_{j_{k+1}}) - g_{j_k}(y_{j_{k+1}})|$$

for each $k \in I$. By $g_{j_{k+1}}(y_{j_{k+1}}) > 3\varepsilon$ and (1) we obtain

$$|g_{j_{k+1}}(x) - g_{j_k}(x)| > \varepsilon$$

for each $k \in I$. A contradiction with the fact that (g_{j_k}) is a Cauchy sequence on X_1. So $f(x_n) \to 0$.

In a specially important case, when X is a normed space, we obtain by Proposition 3.2 and Theorem 3.3 the classical Orlicz-Pettis Theorem and also the Orlicz-Pettis type theorems II.3 and II.4 from [9].

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Received by the editors April 11, 1984.

REZIME

JEDNA BELEŠKA O NEKIM KONVERGENCIJAMA NAD POLUGRUPAMA

U radu se uvodi novi pojam H-Košijev nizovni uslov, te se pomoću njega dokazuje jedna opšta teorema o konvergenciji u komutativnoj polugrupi. Neka je X komutativna polugrupa sa neutralnim elementom koja je snabdevena netrivijalnom subaditivnom i homogenom funkcionelom f. Neka je H familija konačnih aditivnih funkcionela h nad X sa osobinom $h(x) \leq f(x)$ $(x \in X,\ h \in H)$. Kažemo da X zadovoljava H-Košijev nizovni uslov ako za svaki niz (y_j) iz X takav da je

$$\sum_{j=1}^{\infty} |h(y_j)| < \infty \qquad (h \in H)$$

i svaki niz (h_n) iz H takav da je Košijev niz nad (y_j), tada je (h_n) Košijev niz i nad $((y_j),H)$-polugrupom X_1 ($x \in X_1$ ako i samo ako $h(u_1+\ldots+u_k) \to h(x)$ za $k \to \infty$ i sve $h \in H$ za neki niz (u_j) takav da je u_j ili $\lambda_j y_j$ (za $\lambda_j \in N$) ili O i $\sum_{j=1}^{\infty} |h(u_j)| < \infty$ $(h \in H)$).

Ako je X konačna polugrupa ili je H konačna familija tada X uvek zadovoljava H-Košijev nizovni uslov. U slučaju normiranog vektorskog prostora X, X zadovoljava B^*-Košijev nizovni uslov (B^* je jedinična lopta u dualu X^*)- Propozicija 2.1.

Za familiju H se kaže da zadovoljava ε-uslov ako za svako $x_o \in X$, svako $\varepsilon > 0$ i svaku aditivnu funkcionelu h' nad X sa osobinom $h(x) \leq f(x)$ $(x \in X)$ postoji $h \in H$ tako da je

$$h(x_o) + \varepsilon > h'(x_o).$$

U glavnoj teoremi 3.3 se dokazuje da ako niz (x_n) iz polugrupe X, koja zadovoljava H-Košijev nizovni uslov za familiju H koja zadovoljava ε-uslov, ima osobinu da za svaki njegov podniz (y_n) postoji $y \in X$ tako da je

$$h(y_1+\ldots+y_n) \to h(y) \quad \text{kada } n \to \infty \quad (h \in H),$$

tada $f(x_n) \to 0$ za $n \to \infty$.

Pomoću ove teoreme se dokazuju neke teoreme tipa Orlicz-Pettisa.