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ON A CLASS OF SPACES OF THE TYPE $S'\{M_p(x,q)\}$

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ABSTRACT

We analyze the structure of the space $\sigma\{M_p\}$ and $\sigma'\{M_p\}$. Under certain conditions on the matrix $\{C_{p,q}\cdot\exp(m_p(x))\}$ we investigate relations between the space $\sigma'\{M_p\}$ and some spaces of ultradistributions. Also we investigate the Fourier transformation on the spaces $\sigma\{M_p\}$ and $\sigma'\{M_p\}$.

1. INTRODUCTION

The spaces of the type $S'\{M_p(x,q)\}$ were introduced in [10], though some examples of such spaces were analyzed already in [1]. In [9] a class of spaces of the type $S'\{M_p(x,q)\}$ was investigated.

In this paper we shall observe a class of spaces of the type $S'\{M_p(x,q)\}$ denoted by $\sigma'\{M_p(x,q)\}$ (short. $\sigma'\{M_p\}$ or σ') for $M_p(x,q) = C_{p,q}\exp(m_p(x))$, $(p,q)\in \mathbb{N}\times\mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Throughout the paper $\{C_{p,q};\ (p,q)\in\mathbb{N}\times\mathbb{N}_0\}$ (short. $\{C_{p,q}\}$) denotes an infinite matrix of positive numbers and $\{m_p(x),\ p\in N\}$ (short. $\{m_p(x)\}$) denotes a sequence of functions. The properties of $\{C_{p,q}\}$ and $\{m_p(x)\}$ will be given later.

We shall analyze the structure of the spaces σ and σ'. The elements of σ' we shall call "exponential ultradistributions". Particularly, we shall prove that under certain conditions

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the space of test functions $\sigma\{M_p\}$ is sufficiently rich and that $\sigma'\{M_p\}$ is a subspace of the space of ultradistributions $\mathcal{D}'^{(N_q)}$ ([3]) for a corresponding sequence $\{N_q\ ;\ q \in \mathbb{N}_o\}$. We shall obtain a representation theorem for exponential ultradistributions. As well, we shall define the space of entire analytic functions on the complex plane which is the Fourier transformation of the space $\sigma\{M_p\}$. This will enable us to define the Fourier transformation of exponential ultradistributions.

2. SPACES $\sigma\{M_p\}$ AND $\sigma'\{M_p\}$

Let $\{C_{p,q}\}$ be an infinite matrix with positive numbers. For this matrix we suppose:

(C.1) $C_{p,q} \le C_{p+1,q}$ for every $(p,q) \in \mathbb{N} \times \mathbb{N}_o$;

(C.2) For every $p \in \mathbb{N}$ the sequence $\{C_{p,q},\ q \in \mathbb{N}_o\}$ monotonically tends to zero when $q \to \infty$.

(C.3) For every $p \in \mathbb{N}$ there exists $p' \in \mathbb{N}$, $p' > p$, such that for every $\varepsilon > 0$ there exists $q_o(\varepsilon) \in \mathbb{N}_o$ with the property $C_{p,q} \le \varepsilon C_{p',q}$ for $q \ge q_o(\varepsilon)$.

(C.3) makes that (C.1) is superfluous in the theory of spaces σ an σ'. We assume that (C.1) holds only to make the whole theory easier:

In order to have the differentiation as an inner operation in σ' we shall suppose as well:

(C.4) For every $p \in \mathbb{N}$ there exists $p' \in \mathbb{N}$, such that $\sup\{C_{p,q}/C_{p',q+1}\ ;\ q \in \mathbb{N}_o\} < \infty$.

Let $\{\mu_p(t);\ p \in \mathbb{N}\}$, $t \ge 0$, be a sequence of continuous increasing functions which satisfy: $\mu_p(0) = 0$, $\mu_p(\infty) = \infty$ and $\mu_p(t) \le \mu_{p+1}(t)$ for every $t \ge 0$, $p \in \mathbb{N}$. Putting

$$m_p(t) = \int_0^{|t|} \mu_p(u)du,\ p \in \mathbb{N},\ t \in \mathbb{R},$$

we obtain another sequence of functions. Every $m_p(t)$, $p \in \mathbb{N}$, is an even convex function which increases to infinity faster than any linear function when $|t| \to \infty$. This implies that its dual function in the sense of Young

$$\tilde{m}_p(y) := \int_0^{|y|} \mu_p^{-1}(t)dt = \sup\{|t\cdot y| - m_p(t);\ t \in \mathbb{R}\}$$

is finite for arbitrary $y \in \mathbb{R}$; $\mu_p^{-1}(t)$, $t \geq 0$ is the inverse function of $\mu_p(t)$ (see [2]). We suppose also in the sequel that the following condition (introduced in [6]), holds :

(A) For every $p \in \mathbb{N}$ there exists $p' \in \mathbb{N}$ such that

$$m_p(pt) \leq m_{p'}(t) \quad \text{holds for} \quad |t| \geq p'.$$

We denote by $\{m_{p,1}(x_1)\}, \ldots, \{m_{p,s}(x_s)\}$, $p \in \mathbb{N}$, the sequences of functions obtained from the corresponding sequences $\{\mu_{p,1}(x_1)\}, \ldots, \{\mu_{p,s}(x_s)\}$ in the above construction, and we put

$$m_p(x) = m_{p,1}(x_1) + \ldots + m_{p,s}(x_s) \ , \quad x = (x_1, \ldots, x_s) \in \mathbb{R}^s,$$

Since the sequences $\{m_{p,i}(x_i)\}$, $i = 1, \ldots, s$ satisfy (A), this condition (in an obvious interpretation) holds also for $\{m_p(x)\}$.

Further on in the paper we shall put

$$M_p(x, q) = C_{p,q} \exp(m_p(x)),\ p \in \mathbb{N},\ q \in \mathbb{N}_o,\ x \in \mathbb{R}^s .$$

DEFINITION 1. *The vector space of smooth functions on* $\mathbb{R}$ *such that for every* $p \in \mathbb{N}$

$$\gamma_p(\phi) := \sup\{|\phi^{(q)}(x)| M_p(x, |q|);\ x \in \mathbb{R}^s,\ q \in \mathbb{N}_o^s\} < \infty$$

is denoted by $\sigma\{M_p(x, q)\}$ *(short.* $\sigma\{M_p\}$*) . The topology in the space* $\sigma\{M_p\}$ *is given by the sequence of norms* $\{\gamma_p;\ p \in \mathbb{N}\}$. *(As usual,* $|q| = q_1 + \ldots + q_s$ *where* $q = (q_1, \ldots, q_s)$.*).*

In the usual manner (see [1]) one checks that a sequence $\{\phi_n(x)\}$ from $\sigma\{M_p\}$ converges to $\phi \in \sigma\{M_p\}$ iff on every compact set $K \subset \mathbb{R}$ and every $q \in \mathbb{N}_o^s$ the sequence $\{\phi_n^{(q)};\ n \in \mathbb{N}\}$ converges uniformly to $\phi^{(q)}$ and for every $p \in \mathbb{N}$ there exists $C_p > 0$ such that $\gamma_p(\phi_n) \leq C_p$, for every $n \in \mathbb{N}$.

PROPOSITION 1. *Let* $\phi \in \sigma\{M_p\}$. *Then*

(i) $$\lim_{|q|\to\infty} \sup\{|\phi^{(q)}(x)|M_p(x,|q|);\ x \in \mathbb{R}^s\} = 0$$

(ii) $$\lim_{|x|\to\infty} \sup\{|\phi^{(q)}(x)|M_p(x,|q|);\ q \in \mathbb{N}_o^s\} = 0$$

P r o o f. (i) follows from (C.3) and (ii) follows from the fact that $m_p(x) \to \infty$ if $|x| \to \infty$.

We denote by σ_p, $p \in N$, a subspace of $C^\infty(\mathbb{R}^s)$ such that $\phi \in \sigma_p$ iff

$$\gamma_p(\phi) < \infty,\ \lim_{|q|\to\infty} \sup\{|\phi^{(q)}(x)|M_p(x,|q|);\ x \in \mathbb{R}^s\} = 0 \text{ and}$$

$$\lim_{|x|\to\infty} \sup\{|\phi^{(q)}(x)|M_p(x,|q|);\ q \in \mathbb{N}_o^s\} = 0.$$

THEOREM 1. (i) *The space* σ_p *is a Banach space.*
(ii) *The space* $\sigma\{M_p\}$ *is a Frechet-Schwartz space.*

P r o o f. (i) Let $\gamma_p(\phi_\nu - \phi_\mu) < \varepsilon$ if $\nu,\mu \geq N(\varepsilon)$ and let $\phi \in C^\infty(\mathbb{R}^s)$ be the limit of the sequence $\{\phi_\nu\}$.

We prove that $\phi \in \sigma_p$. Clearly $\gamma_p(\phi) < \infty$ holds. We want to prove the remaining properties of ϕ. First we prove that for every $q \in \mathbb{N}_o^s$

(a) $$\sup\{M_p(x,|q|)|\phi_\nu^{(q)} - \phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} \to 0 \text{ as } \nu \to \infty .$$

If $\nu,\mu_o > N(\varepsilon)$ we have

(b) $$\sup\{M_p(x,|q|)|\phi_\nu^{(q)}(x)|;\ x \in \mathbb{R}^s \setminus K\} \leq$$
$$\leq \sup\{M_p(x,|q|)|\phi_{\mu_o}^{(q)}(x)|;\ x \in \mathbb{R}^s \setminus K\} + \varepsilon ,$$

where $K = B(0,\rho)$ is the closed ball with radius $\rho > 0$. If $\nu \to \infty$ we obtain

$$\sup\{M_p(x,|q|)|\phi^{(q)}(x)|;\ x \in \mathbb{R}^s \setminus K\} \leq$$
$$\leq \sup\{M_p(x,|q|)|\phi_{\mu_o}^{(q)}(x)|;\ x \in \mathbb{R}^s \setminus K\} + \varepsilon .$$

For every compact set $K \subset \mathbb{R}^s$ there exists $N(\varepsilon,K)$ such that

$$\sup\{M_p(x,|q|)\,|\phi_\nu^{(q)}(x) - \phi^{(q)}(x)|;\ x \in K\} < \varepsilon \text{ if } \nu > N(\varepsilon,K).$$

because $\phi_\nu^{(q)}$ converges uniformly to $\phi^{(q)}$ on K.

Let $\mu_o > N(\varepsilon)$ and let ρ be chosen such that for $K = B(0,\rho)$ the following estimate holds

$$\sup\{M_p(x,|q|)|\phi_{\mu_o}^{(q)}(x)|;\ x \in \mathbb{R}^s \setminus K\} < \varepsilon \quad .$$

Therefore taking $\nu \geq \nu_o(q) = \max\{N(\varepsilon),\ N(\varepsilon,B(0,\rho))\}$ we have

$$\sup\{M_p(x,|q|)\,|\phi_\nu^{(q)}(x) - \phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} \leq$$

$$\leq \sup\{M_p(x,|q|)\,|\phi_\nu^{(q)}(x) - \phi^{(q)}(x)|;\ x \in K\} +$$

$$+ \sup\{M_p(x,|q|)\,(|\phi_\nu^{(q)}(x)| + |\phi^{(q)}(x)|);\ x \in \mathbb{R}^s \setminus K\} \leq$$

$$\leq \varepsilon + 2\varepsilon + 2\varepsilon = 5\varepsilon \ .$$

Thus we proved (a).

Since $\phi_{\mu_o} \in \sigma_p$, there exists $N_o(\varepsilon)$ such that

$$\sup\{M_p(x,|q|)|\phi_{\mu_o}^{(q)}(x)|;\ x \in \mathbb{R}^s\} < \varepsilon \quad \text{if} \quad |q| > N_o(\varepsilon) \ .$$

From (b) we obtain that $\nu > N(\varepsilon)$ and $|q| > N_o(\varepsilon)$ imply

$$\sup\{M_p(x,|q|)\,|\phi_\nu^{(q)}(x)|;\ x \in \mathbb{R}^s\} < 2\varepsilon \ .$$

For a fixed $q \in N_o^s$ and $\nu(q) > \nu_o(q)$ we have

$$\sup\{M_p(x,|q|)\,|\phi^{(q)}(x) - \phi_{\nu(q)}^{(q)}(x)|;\ x \in \mathbb{R}^s\} < \varepsilon \ .$$

Thus, from

$$\sup\{M_p(x,|q|)\,|\phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} \leq$$

$$\leq \sup\{M_p(x,|q|)\,|\phi^{(q)}(x) - \phi_{\nu(q)}^{(q)}(x)|;\ x \in \mathbb{R}^s\} +$$

$$+ \sup\{M_p(x,|q|)|\phi_{\nu(q)}^{(q)}(x)|;\ x \in \mathbb{R}^s\}$$

we obtain that

$$\lim_{|q|\to\infty} \sup\{M_p(x,|q|)|\phi^{(q)}(x)|;\ x\in \mathbb{R}^s\} = 0 .$$

The proof of

$$\lim_{|x|\to\infty}\{\sup M_p(x,|q|)|\phi^{(q)}(x)|;\ q\in \mathbb{N}_o^s\} = 0$$

may be derived in a similar way by observing separately this supremum for $|q|\le q_o$ and $|q|>q_o$ for a suitable $q_o\in\mathbb{N}_o$.

(ii) Proposition 1 implies that $\sigma\{M_p\} = \bigcap_{p=1}^{\infty}\sigma_p$.

Let p" be an integer such that $p''\ge p'$ where p' is an integer which corresponds to given $p\in\mathbb{N}$ in condition (C.3). From condition (A) it follows that we may choose p" such that $\exp(m_p(x)-m_{p''}(x))\to 0$ as $|x|\to\infty$. We shall show that the inclusion mapping $\sigma_{p''}\to\sigma_p$ is compact. For the proof we shall use an idea from [1] .

Let $\{\phi_\nu\}$ be a bounded sequence in $\sigma_{p''}$. We denumerate the set $\mathbb{N}_o^s$ by putting $e_1=(1,\ ,\dots,0)\to 1$, $e_2=(0,1,\dots,0)\to 2,\dots$ etc. By $\{K_n\}$ we denote a sequence of compact subsets of $\mathbb{R}^s$ such that

$$K_n\subset \overset{o}{K}_{n+1},\ n\in\mathbb{N},\quad \bigcup_{n=o}^{\infty} K_n = \mathbb{R}^s \quad\text{and}$$

$$\sup\{\exp(m_p(x)-m_{p''}(x));\ x\in\mathbb{R}^s\setminus K_n\}<\varepsilon_n$$

where the sequence $\{\varepsilon_n\}$, $\varepsilon_n>0$, monotonically tends to zero ($\overset{o}{K}_n$ is the interior of K_n).

The functions $|\phi_\nu^{(e_1)}(x)|$, $\nu\in\mathbb{N}$, are uniformly bounded on K_1. Hence, by virtue of the Arzela theorem, there exists a sub-sequence $\{\phi_{\nu,1}\}$ of $\{\phi_\nu\}$ which converges uniformly on K_1. Because of the uniform boundedness of the functions $|\phi_{\nu,1}^{(e_1)}(x)|$, $\nu,1=\nu_1\in\mathbb{N}$, on K_2 , according to the same Arzela theorem there exists a sub-sequence $\{\phi_{\nu,2}\}$ of $\{\phi_{\nu,1}\}$ such that $\{\phi_{\nu,2}^{(e_2)}\}$ converges uniformly on K_2. Continuing in this manner, and then applying a diagonalization process we obtain a sequence $\{\phi_{\nu,\nu}\}$.

As $\{\phi_{\nu,\nu}^{(q)}\}$ converges to $\phi^{(q)}$ on every compact set $K \subset \mathbb{R}^s$ and $\gamma_{p'}(\phi_{\nu,\nu}) \le M$ we have that $\gamma_{p''}(\phi) \le M$ because of

$$|\phi_{\nu,\nu}^{(q)}(x)| \le (M / C_{p'',|q|})\exp(-m_{p''}(x))$$

which holds on every compact set $K \subset \mathbb{R}^s$.

We shall show that $\phi \in \sigma_p$ and that $\{\phi_{\nu,\nu}\}$ converges to ϕ in σ_p.

For a fixed $q \in \mathbb{N}_o^s$ we have

(c) $$\sup\{C_{p,|q|} \exp(m_p(x))|\phi^{(q)}(x)|;\ x \in \mathbb{R}^s \setminus K_n\} \le$$

$$\le \varepsilon_n (C_{p,|q|} / C_{p'',|q|}) \sup\{C_{p'',|q|} \exp(m_{p''}(x))|\phi^{(q)}(x)|;$$
$$x \in \mathbb{R}^s \setminus K_n\} \le \varepsilon_n .$$

where ε_n is from (C.3).

Let $\{\varepsilon_n'\}$ be a sequence of real numbers which tends to zero and let $q_o(\varepsilon_n')$, $n \in \mathbb{N}$, be the corresponding numbers from condition (C.3). From the inequality

$$\sup\{C_{p,|q|} \exp(m_p(x))|\phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} \le$$
$$\le \varepsilon_n' \cdot \exp(m_p(x) - m_{p''}(x)) \sup\{C_{p'',|q|} \exp(m_{p''}(x))|\phi^{(q)}(x)|;$$
$$x \in \mathbb{R}^s\} \le \varepsilon_n' M$$

which holds for $|q| > q_o(\varepsilon_n')$, we obtain that

$$\lim_{|q|\to\infty} \sup\{C_{p,|q|} \exp(m_p(x))|\phi^{(q)}(x)|;\ x \in \mathbb{R}^s\} = 0.$$

This fact, together with (c), implies

$$\lim_{|x|\to\infty} \sup\{C_{p,|q|}|\phi^{(q)}(x)|\exp(m_p(x));\ q \in \mathbb{N}_o^s\} = 0.$$

So we proved that $\phi \in \sigma_p$.

We have

$$\gamma_p(\phi_{\nu\nu}-\phi) \leq \sup\{M_p(x,|q|)\,|\phi_{\nu\nu}^{(q)}(x)-\phi^{(q)}(x)|;\ x\in K_n,\ q\in \mathbb{N}_o^s\}+$$

$$+\sup\{M_p(x,|q|)\,|\phi_{\nu\nu}^{(q)}(x)-\phi^{(q)}(x)|;\ x\in \mathbb{R}^s\setminus K_n,\ q\in \mathbb{N}_o^s\}\leq$$

$$\leq \sup\{M_p(x,|q|)\,|\phi_{\nu\nu}^{(q)}(x)-\phi^{(q)}(x)|;\ x\in K_n,\ |q|\leq q_o(\varepsilon_n')\}+$$

$$+\sup\{M_p(x,|q|)(|\phi_{\nu\nu}^{(q)}(x)|+|\phi^{(q)}(x)|);\ x\in K_n,\ |q|>q_o(\varepsilon_n')\}+$$

$$+\sup\{M_p(x,|q|)(|\phi_{\nu\nu}^{(q)}(x)|+|\phi^{(q)}(x)|);\ x\in \mathbb{R}^s\setminus K_n, |q|\leq$$

$$\leq q_o(\varepsilon_n')\}+\sup\{M_p(x,|q|)(|\phi_{\nu\nu}^{(q)}(x)|+|\phi^{(q)}(x)|);$$

$$x\in \mathbb{R}^s\setminus K_n,\ |q|>q_o(\varepsilon_n')\}\leq \sup\{M_p(x,|q|)\,|\phi_{\nu\nu}^{(q)}(x)-$$

$$-\phi^{(q)}|;\ x\in K_n,\ |q|\leq q_o(\varepsilon_n')\}+\varepsilon_n' 2M+\varepsilon_n 2M+\varepsilon_n'\varepsilon_n 2M.$$

Therefore from the construction of the sequence $\{\phi_{\nu\nu}(x)\}$ it follows that $\{\phi_{\nu\nu}\}$ converges to ϕ in σ_p.

We shall turn now to an important example of the space $\sigma\{M_p\}$.

3. IMBEDDING OF $\sigma'\{M_p\}$ INTO ULTRADISTRIBUTIONS

Let

$$(1)\qquad c_{p,q}=\frac{p^q}{N_q}$$

where $\{N_q;\ q\in N_o\}$ is an increasing sequence of positive numbers such that the following conditions holds:

(M.1) $\quad N_q^2\leq N_{q-1}N_{q+1}\ ,\quad q\in \mathbb{N}_o;$

(M.2)' There are $A>0$ and $H>0$ such that

$$N_q\leq AH^qN_{q+1}\ ,\quad q\in N_o\ ;$$

(M.3)' $\quad \sum_{q=o}^{\infty} N_q/N_{q+1}<\infty\ .$

(see [3]). Observe that then the matrix $\{\frac{p^q}{N_q}\}$ satisfies the conditions (C.1)-(C.4). Now putting $M_p(x,|q|) = \frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))$ we come to an example of a space of the type $\sigma\{M_p\}$, i.e. to the space $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$; of course, here $q \in \mathbb{N}_o^s$, $x \in \mathbb{R}^s$.

One checks easily that the space $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$ (see [4]) is a subspace of $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$, in fact we shall prove something more:

THEOREM 2. a) *The space* $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$ *is a dense subspace of* $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$.

b) *The space* $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$ *is sufficiently rich in the sense of* [1].

P r o o f. a) Let $\phi(x)$ be an arbitrary function from $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$ (short. $\sigma_1\{M_p\}$). We shall construct a sequence of functions from $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$, which converges to $\phi(x)$ in $\sigma_1\{M_p\}$.

First, let us recall that condition (M.3)′ implies that there exists a nonnegative function with compact support $h(x)$ from $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$ such that $h(x) \equiv 1$ on the interval $[-1,1]^s$.

Let $h_n(x) := h(\frac{x}{n})$, $n \in \mathbb{N}$; obviously $h_n(x) \in \mathcal{D}^{(N_q)}(\mathbb{R}^s)$ for every $n \in \mathbb{N}$ and let us put $K_n := \operatorname{supp} h_n$.

We shall use the following inequality

$$(2) \qquad \sup\{\frac{p^{|q|}}{N_{|q|}}|h_n^{(q)}(x)|;\ x \in \mathbb{R}^s\} \le \frac{1}{n^{|q|}} \sup\{\frac{p^{|q|}}{N_{|q|}}|h^{(q)}(x)|;\ x \in \mathbb{R}^s\}$$

for every $p,n \in \mathbb{N}$, $q \in \mathbb{N}_o^s$.

We prove now that the sequence $\{h_n(x)\phi(x)\}$ converges to $\phi(x)$ in the sense of $\sigma_1\{M_p\}$. It is clear that for every compact set $K\subset \mathbb{R}^s$ the sequence $\{(h_n(x)\phi(x))^{(q)};\ n\in\mathbb{N}\}$ converges uniformly to $\phi^{(q)}(x)$ on K. So, we have yet to prove that for every $p\in\mathbb{N}$ there exists $C_p>0$ with the property

(3) $$\gamma_p(h_n(x)\phi(x)) \le C_p .$$

In fact, from the inequality $N_r N_{q-r} \le N_o N_q$, $0\le r\le q$, $r,q\in\mathbb{N}_o$, which follows from (M.1), we have

$$\sup\{|(h_n(x)\phi(x))^{(q)}|\exp(m_p(x));\ x\in\mathbb{R}^s\} \le$$

$$\le \sum_{r\le q}\binom{q}{r}\sup\{|h_n^{(r)}(x)|;\ x\in K_n\}\sup\{|\phi^{(q-r)}(x)|\cdot$$

$$\cdot\exp(m_{2p}(x));\ x\in K_n\} \le \sum_{r\le q}\binom{q}{r}\frac{N_{|r|}N_{|q-r|}}{(2p)^{|q-r|}}\sup\{|h_n^{(r)}(x)|\cdot$$

$$\cdot\frac{2p^{|r|}}{N_{|r|}};\ x\in K_n\}\gamma_{2p}(\phi) \le \frac{N_o N_{|q|}}{(2p)^{|q|}}\sum_{r\le q}\binom{q}{r}\sup\{|h^{(q)}(x)|\cdot$$

$$\cdot\frac{(2p)^q}{N_q};\ x\in\mathbb{R},\ q\in\mathbb{N}_o\}\gamma_{2p}(\phi) \le C_p\frac{N_{|q|}}{p^{|q|}}$$

for some $C_p\ge 0$ (as usual, $\binom{q}{r}=\binom{q_1}{r_1}\dots\binom{q_s}{r_s}$, $r\le q \iff r_i\le q_i$, $i=1,2,\dots,s$).

b) We shall check all three conditions from the Lemma on page 236 in [1]. We already know that there exists a nontrivial function in $\sigma_1\{M_p\}$. The translation-invariance of the space $\sigma_1\{M_p\}$ follows from condition (A). In fact, by (A) for given $t\in\mathbb{R}^s$ and $p\in\mathbb{N}$ there exists a $p'\in\mathbb{N}$ such that $m_p(x)\le m_{p'}(x-t)$ for $|x|$ sufficiently large. For $\phi\in\sigma_1\{M_p\}$, this implies

$$\gamma_p(\phi(x-t)) = \sup\{|\phi^{(q)}(x-t)|\frac{p^{|q|}}{N_{|q|}} \exp(m_p(x));\ x \in \mathbb{R}^s, q \in \mathbb{N}_o^s\}$$

$$\leq C\cdot\sup\{|\phi^{(q)}(x)|\frac{p^{|q|}}{N_{|q|}} \exp(m_{p'}(x));\ x \in \mathbb{R}^s,\ q \in \mathbb{N}_o^s\} < \infty$$

for some $C > 0$ which does not depend on x and q. At last, we must show that for arbitrary $t \in \mathbb{R}^s$, the function $\phi(x)\exp i(x,t)$ is in $\sigma_1\{M_p\}$, provided that $\phi \in \sigma_1\{M_p\}$. (As usual, $(x,t) \doteq x_1t_1 + \dots + x_st_s$). We have

$$\sup\{|D^q(\phi(x)\exp i(x,t))|\exp(m_p(x));\ x \in \mathbb{R}^s\} \leq$$

$$\leq \sum_{r\leq q} \binom{q}{r} \sup\{|\phi^{(q-r)}(x)|\frac{(2p)^{|q-r|}}{N_{|q-r|}} \exp(m_{2p}(x));\ x \in \mathbb{R}^s\}$$

$$\cdot \frac{(2p|t|)^{|q|}}{(2p)^{|q|}N_{|r|}} N_{|r|}N_{|q-r|} \leq \gamma_{2p}(\phi)\frac{N_oN_q}{(2p)^q}\sum_{r\leq q}\binom{q}{r}\frac{(2p|t|)}{N_{|r|}} \leq$$

$$\leq C_1 \sup\{\frac{|2pt|^{|q|}}{N_{|q|}};\ q \in \mathbb{N}_o^s\}\cdot \frac{N_{|q|}}{p^{|q|}} \leq C_2 \cdot \frac{N_{|q|}}{p^{|q|}} < \infty$$

since $\sup\{\frac{|2pt|^{|q|}}{N_{|q|}};\ q \in N_o^s\} < \infty$ in view of (M.3)'; C_1 and C_2 are positive constants which do not depend on x or q.

This theorem shows that $\sigma'\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\}$ (the dual of the space $\sigma\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p))\}$) is a subspace of a space of ultradistributions $\mathcal{D}^{(N_q)'}(\mathbb{R}^s)$.

A sufficient condition which implies that the space $\sigma'\{M_p\}$ is a subspace of a space of ultradistributions is given in the following theorem. Its proof is similar to that of Theorem 2, so we omit it.

THEOREM 3. *Let us suppose for the matrix* $\{C_{p,q}\}$ *that the following condition holds as well:*

(4) *For every* $p \in \mathbb{N}$ *there exist* $p' \in \mathbb{N}$, $p' > p$, *and* $K_{p,p'} > 0$ *such that*

$$2^q C_{p,q} \leq K_{p,p'} \cdot C_{p',q-r} \cdot \frac{(p')^r}{N_r}, \quad r,q \in \mathbb{N}_o, \quad r \leq q.$$

If $\{N_q\}$ *satisfies* (M.1) *and* (M.3)′, *then*.

a) *The space* $\mathcal{D}^{(N_q)}(\mathbb{R}^s)$ *is a dense subspace of the space* $\sigma\{M_p\}$;

b) *The space* $\sigma\{M_p\}$ *is sufficiently rich (in the sense of* [1])

4. A STRUCTURAL THEOREM FOR $\sigma'\{M_p\}$

It is proved in [9] that a linear functional f on $S\{M_p(x, q)\}$ is continuous iff there exists a $p \in \mathbb{N}$ and a sequence of measures $\{f_q;\ q \in \mathbb{N}_o^s\}$ on $\mathbb{R}^s$ such that

$$(5) \qquad \sum_{q \in \mathbb{N}_o^s} (\text{total variation of } f_q) < \infty$$

and for every $\phi \in S\{M_p\}$

$$(6) \qquad \langle f,\phi\rangle = \sum_{q \in \mathbb{N}_o^s} (-1)^{|q|} \cdot \int_{\mathbb{R}^s} M_p(x,|q|)\cdot\phi^{(q)}(x)\,df_q \ .$$

Yamanaka proved this theorem under conditions which are all satisfied in the case $M_p(x,|q|) = C_{p,|q|}\exp(m_p(x))$; i.e., the representation (6), under the convergence of the series (5), is valid for the elements from $\sigma'\{M_p\}$. However, for this space we shall obtain a somewhat more precise structural theorem.

THEOREM 4. *A linear functional* f *on* $\sigma\{M_p\}$ *is continuous iff there exists a* $p \in \mathbb{N}$ *and a sequence of functions from* $L_\infty^{loc}(\mathbb{R}^s)$: $\{f_q(x);\ q \in \mathbb{N}_o^s\}$ *such that*

$$(5') \qquad \sum_{q\in \mathbb{N}_o^s} \operatorname{ess\,sup}\{|f_q(x)|;\ x \in \mathbb{R}^s\} < \infty$$

and for every $\phi \in \sigma\{M_p\}$

$$(6') \qquad \langle f,\phi\rangle = \sum_{q\in \mathbb{N}_o^s} C_{p,q}(-1)^{|q|}\cdot \int_{\mathbb{R}^s} \exp(m_p(x))\cdot\phi^{(q)}(x)\cdot f_q(x)dx .$$

P r o o f. First we have to prove that the sequence of norms $\{\gamma_p\}$ is equivalent to the sequence of norms $\{\eta_p\}$ where

$$\eta_p(\phi) := \sup\{\int_{\mathbb{R}^s} M_p(x,|q|)|\phi^{(q)}(x)|dx;\ q\in\mathbb{N}_o^s\},\ p\in\mathbb{N} .$$

In [6] we proved that condition (A) implies condition (N) ([1]) for every sequence $\{\exp(m_{p,i}(x_i))\}$, $i=1,2,\dots,s$. This fact is crucial for the proof of equivalence of sequences $\{\gamma_p\}$ and $\{\eta_p\}$. In ([7], this Journal) we discuss more about condition (N). Thus, using arguments of the proof of ([7] Lemma 3. (i)) and remarks given in ([7], part 2) one can prove that the sequences $\{\gamma_p\}$ and $\{\eta_p\}$ are equivalent.

For a fixed $p\in N$ we denote by σ_{1p} the normed space defined in the following way

$$\phi\in\sigma_{1p} \text{ iff } \phi\in C^\infty(\mathbb{R}^s)\mu_p(\phi)<\infty \text{ and}$$

$$\lim_{|q|\to\infty}\int_{R^s} M_p(x,|q|)|\phi^{(q)}(x)|dx = 0 ,$$

Similarly to the proof of Theorem 1 (i) one can prove that σ_{1p}, $p\in\mathbb{N}$, are (B) spaces, $\sigma_{11}\supset\sigma_{12}\supset\dots$ and that the norms $\{\eta_p\}$ are pairwisely compatible. If we denote by σ^p the completion of the space σ according to the norm η_p, $p\in\mathbb{N}$, from [1] p.35 we obtain

$$\sigma' = \bigcup_{p=1}^{\infty}(\sigma^p)'$$

It means that any element f from σ' may be extended from the space σ onto the space σ^p (for some p); this element from $(\sigma^p)'$ let us denote also by f. The σ^p is a closed subspace of the space σ_{1p}. By Hahn-Banach Theorem f may be continuously extended from σ^p on σ_{1p} to be continuous. Contrary, a restriction of any element from σ'_{1p} on σ^p belongs to $(\sigma^p)'$. Since we want to give a representation theorem for the elements from σ', by the given explanations it is enough to prove a representation theorem for elements from σ'_{1p}.

We denote by Γ the subspace of $\prod_{q\in N_0^s} L^1(\mathbb{R}^s)$ defined in the following way

$$\psi = (\phi_q) \in \Gamma \text{ iff } \quad ||\psi|| := \sup\{ \int_{\mathbb{R}^s} |\phi_q(x)|dx;\ q \in \mathbb{N}_0^s \} < \infty$$

$$\text{and } \lim_{|q|\to\infty} \int_{\mathbb{R}^s} |\phi_q(x)|dx = 0 .$$

The space σ_{1p} is isometrically isomorphic to a subspace of Γ, $\Gamma_p = u(\sigma_{1p})$, where u is the mapping defined in the following way

$$\sigma_{1p} \ni \phi \to u(\phi) = (M_p(x,|q|)\cdot \phi^{(q)}(x)) \in \Gamma_p \quad .$$

If $f \in \sigma'_{1p}$ then by

$$<\tilde{f},\psi> := <f,u^{-1}(\psi)> \ , \quad \psi \in \Gamma_p \ ,$$

an element from Γ'_p is defined. By Hahn-Banach Theorem $\tilde{f}$ may be extended on Γ to be an element from Γ'; let us denote this element by F. It is known (see [9] or [4]) that if $F \in \Gamma'$ then there exist functions f_q, $q \in \mathbb{N}_0^s$, from $L^\infty(\mathbb{R}^s)$ such that

$$<F,\psi> = \sum_{q\in\mathbb{N}_0^s} \int_{\mathbb{R}^s} f_q(x)\phi_q(x)dx, \ \psi = (\phi_q) \in \Gamma \quad \text{and}$$

$$\sum_{q\in\mathbb{N}_0^s} \text{ess sup}\{|f_q(x)|;\ x \in \mathbb{R}^s \} < \infty$$

It means that on σ_{1p} we have

$$<f,\phi> = <\tilde{f},u(\phi)> = \sum_{q\in \mathbb{N}_o^s} \int_{\mathbb{R}^s} f_q(x)M_p(x,|q|)\phi^{(q)}(x)dx =$$

$$= < \sum_{q\in \mathbb{N}_o^s} (-1)^{|q|}(f_q(x)M_p(x,|q|))^{(q)}, \phi(x)> .$$

We obtain that $f \in \sigma'_{1p}$ iff f is of the form

$$(7) \qquad f = \sum_{q\in \mathbb{N}_o^s} (-1)^{|q|}(f_q(x)M_p(x,|q|))^{(q)}$$

such that

$$(8) \quad \sum_{q\in \mathbb{N}_o^s} \text{ess sup}\{|f_q(x)|;\ x\in \mathbb{R}^s\} < \infty$$

where the series in (7) converges weakly in σ_{1p}.

Let us prove now a more suitable representation theorem.

THEOREM 5. *A linear functional f on $\sigma\{M_p\}$ is continuous iff there exist $p_1 \in \mathbb{N}$ and continuous functions $F_q(x)$, $q \in \mathbb{N}_o^s$, on $\mathbb{R}^s$ with the property* $\sum_{q\in \mathbb{N}_o^s} \sup\{|F_q(x)|,\ x\in \mathbb{R}^s\} < \infty$ *such that for every* $\phi \in \sigma\{M_p\}$

$$(9) \quad <f,\phi> = \sum_{q\in \mathbb{N}_o^s} C_{p_1,|q|}(-1)^{|q|}\int_{\mathbb{R}^s} \exp(m_{p_1}(x))\phi^{(q)}(x)F_q(x)dx$$

P r o o f. By what was said before, the condition is sufficient. Conversely, from representation (6´) we obtain that there exist a natural number p and bounded measurable functions f_q such that (5´) holds with the property

$$<f,\phi> = \sum_{q\in \mathbb{N}_o^s} C_{p,|q|}(-1)^{|q|}\int_{\mathbb{R}^s} \exp(m_p(x))\phi^{(q)}(x)f_q(x)dx$$

$\phi \in \sigma\{M_p\}$, or symbolically

$$(10) \qquad f = \sum_{q \in \mathbb{N}_0^s} C_{p,|q|} D^q(\exp(m_p(x)) f_q(x)).$$

Let us choose $p_1 \in \mathbb{N}$ such that the function $x \cdot \exp(m_p(x) - m_{p_1}(x))$ is bounded (see (N)) and condition (C.4) holds. Then we obtain

$$(10)' \quad f = \sum_{q \in \mathbb{N}_0^s} C_{p_1,|q+1|} D^{q+1}(\exp(m_{p_1}(x) F_{q+1}(x)))$$

where

$$F_{q+1}(x) := \frac{C_{p,|q|}}{C_{p_1,|q+1|}} \exp(-m_{p_1}(x)) \int_0^x \exp(m_p(t)) f_q(t)dt, \ q \in \mathbb{N}^s$$

are bounded continuous functions or $\mathbb{R}^s$. Since

$$\sum_{q \in \mathbb{N}_0^s} \sup\{|F_{q+1}(x)|; \ x \in \mathbb{R}^s\} \le \sup\{\frac{C_{p,q}}{C_{p,|q+1|}}; \ q \in \mathbb{N}_0\} \cdot$$

$$\sup\{|x|\exp(m_p(x) - m_{p_1}(x)); \ x \in \mathbb{R}^s\} \cdot \sum_{q \in \mathbb{N}_0^s} \text{ess sup}\{|f_q(x)|;$$

$$x \in \mathbb{R}^s\} < \infty ,$$

the relation (10)' is the desired representation of f.

5. FOURIER TRANSFORMATION ON $\sigma\{M_p\}$

In this section we define the space of entire analytic functions Ψ such that $F(\phi) = \Psi$ for some $\phi \in \sigma\{M_p\}$; as usual F stands for the Fourier transform. This enables us to define, through the Parseval equality, the Fourier transform of the elements from $\sigma'\{M_p\}$.

We denote by $\zeta = \xi + i\eta$ the s-dimensional complex variable, $\zeta = (\zeta_1, \dots, \zeta_s)$ where $\zeta_k = \xi_k + i\eta_k \in \mathbb{C}$, $k = 1, \dots, s$. As usual, the scalar product $\langle x,\zeta\rangle$ is $\sum_{k=1}^{s} x_k\zeta_k$ for $x=(x_1,\ldots$ $\ldots,x_s)\in\mathbb{R}^s$ and $\zeta=(\zeta_1,\ldots,\zeta_s)\in\mathbb{C}^s$.

The Fourier transform of a function $\phi\in L^1_{loc}(\mathbb{R}^s)$ is defined by

$$(11)\qquad \hat{\phi}(\zeta) = F(\phi(x))(\zeta) = \int_{\mathbb{R}^s} e^{-i\langle x,\zeta\rangle}\cdot\phi(x)\,dx$$

provided that this integral converges. First we prove a Lemma.

LEMMA 1. *Let* $\phi\in\sigma\{M_p\}$. *Then the integral* (11) *defines an entire analytic function* $\hat{\phi}(\zeta)$ *of* $\zeta=\xi+i\eta\in\mathbb{C}^s$ *such that*

$$(12)\qquad |\zeta^{q}\cdot\hat{\phi}(\zeta)| \le \frac{A_p}{C_{p,|q|}}\exp(\tilde{m}_p(\eta)),\quad p=1,2,\ldots,$$

for some $A_p>0$.

Of course, $\tilde{m}_p(\eta)=\tilde{m}_{p,1}(\eta_1)\ldots\tilde{m}_{p,s}(\eta_s)$, $\eta=(\eta_1,\ldots,\eta_s)$. .

P r o o f. Let us take $\eta_o=(\eta_{o,1},\ldots$ $\ldots,\eta_{o,s})$, $\eta_{o,k}>0$, $k=1,\ldots,s$ and estimate the integral

$$(13)\qquad \Big|(-i)^{|q|}\int_{\mathbb{R}^s} x^{q}\cdot e^{-i\langle x,\zeta\rangle}\phi(x)\,dx\Big| \quad\text{for}\quad |\eta_k|\le\eta_{o,k},$$

$$k=1,\ldots,s \quad\text{and}\quad q=(q_1,\ldots,q_s)\in\mathbb{N}_o^s .$$

Since it is less or equal than

$$C\cdot\prod_{k=1}^{s}\int_{\mathbb{R}}(1+|x_k|)^{q_k}\, e^{x_k\eta_k}\, e^{-m_{p,k}(x_k)}\,dx_k \quad\text{for some } C=C(p,q)>0$$

and

$$m_{p,k}(x_k)\ge 3\cdot|x_k|\eta_{o,k}-A_{p,k}\quad (A_{p,k}>0),\ k=1,\ldots,s$$

we obtain that (13) uniformly converges in any "strip" $\{\xi+i\eta\in\mathbb{C}^s;\ |\eta|\le\eta_o\}$. This implies that we can differentiate under the integral sign arbitrary many times. This means that $\hat{\phi}(\zeta)$ is an entire analytic function on $\mathbb{C}^s$. Let us prove now (12); we observe first that

$$|\zeta^q\hat{\phi}(\zeta)| = |F(\phi^{(q)}(x))(\zeta)| = |\int_{\mathbb{R}^s} e^{-i<x,\zeta>}\phi^{(q)}(x)dx| .$$

For given $p \in \mathbb{N}$ we choose $p' \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^s} \exp(m_p(x) - m_{p'}(x))dx < \infty$$

(condition (N)). Now if $\phi \in \sigma\{M_p\}$ we have

$$|\zeta^q\hat{\phi}(\zeta)| \le \frac{C_1}{C_{p',|q|}} \int_{\mathbb{R}^s} \exp(\sum_{i=1}^{s} |x_i||\eta_i| - m_{p'}(x))dx \le$$

$$\le \frac{C_1}{C_{p',|q|}} \sup\{\exp(\sum_{i=1}^{s} |x_i||\eta_i| - m_p(x)); \ x \in \mathbb{R}^s\} \cdot$$

$$\int_{\mathbb{R}^s} \exp(m_p(x) - m_{p'}(x))dx = \frac{C_2}{C_{p',|q|}} \cdot \exp(\tilde{m}_p(\eta)) \le \frac{C_2}{C_{p,|q|}} \exp(\tilde{m}_p(\eta))$$

for some positive constants C_1, C_2 which depend on p but not on $q \in \mathbb{N}_o^s$.

If $C_{p,|q|} = \frac{p^{|q|}}{N_{|q|}}$, $(p,|q|) \in \mathbb{N} \times \mathbb{N}_o$ where $\{N_{|q|};$ $|q| \in \mathbb{N}_o\}$ satisfies the conditions (M1), (M2)´ and (M3)´, from this lemma and ([3], Lemma 3.3), directly follows that for $\phi \in \sigma\{\frac{p^{|q|}}{N_{|q|}} \exp(m_p(x)\}$ we have the following statement:

For any p there exists $C_p > 0$ such that

$$|\hat{\phi}(\zeta)| \le C_p \cdot \exp\{\tilde{m}_p(\eta) - M(p|\eta|)\} ,$$

where $M(\rho)$, $\rho > 0$, is the associated function to $\{N_q\}$,

$$(14) \quad M(\rho) := \sup\{\log \frac{\rho^{|q|} N_{|o|}}{N_{|q|}}; \ |q| \in \mathbb{N}_o\} \text{ (see [3]).}$$

Let us prove an inequality in the opposite direction.

LEMMA 2. *Let $\Psi(\zeta)$ be an entire analytic function such that*

$$(15) \quad |\zeta^q||\Psi(\zeta)| \le \frac{B_p}{C_{p,|q|}} \exp(\tilde{m}_p(\eta)) \quad \text{for } \zeta = \xi + i\eta \in \mathbb{R}^s ,$$

every $(p,q) \in \mathbb{N} \times \mathbb{N}_o$ *and some* $B_p > 0$. *Then the function* ϕ *defined by*

(16) $$\phi(x) := \int_{\mathbb{R}^s} e^{i<x,\xi>}\psi(\xi)d\xi, \quad x \in \mathbb{R}^s$$

is a smooth function on $\mathbb{R}^s$ *which belongs to* $\sigma\{M_p\}$.

Let us observe that for $C_{p,|q|} = \frac{p^{|q|}}{N_{|q|}}$ the inequality (15) can be written as

(17) $$|\Psi(\zeta)| \le B_p \exp(\tilde{m}_p(\eta) - M(p|\zeta|))$$

provided that (M1), (M2)′ and (M3)′ hold for $\{N_{|q|}, |q| \in \mathbb{N}_o\}$. The function M is defined in (14). So in this case we have a more precise statement.

P r o o f of Lemma 2. The behaviour of $\Psi(\zeta)$ for $|\zeta|$ large implies that this integral defines a smooth function on $\mathbb{R}^s$. Let us take $p \in \mathbb{N}$. First, we replace the real hyperplane $\mathbb{R}^s$ in (16) by the hyperplane

$\mathbb{R}^s + i\eta = \{\xi + i\eta; \xi \in \mathbb{R}^s\}$ (we shall fix $\eta \in \mathbb{R}^s$ later);

in fact, from (15) it follows that

$$\phi(x) = \int_{\mathbb{R}^s} e^{i<x,\xi+i\eta>}\psi(\xi+i\eta)d\xi \quad \text{and}$$

$$\phi^{(q)}(x) = \int_{\mathbb{R}^s} i^{|q|}(\xi+i\eta)^q\psi(\xi+i\eta)d\xi \quad (q = (q_1,\dots,q_s) \in \mathbb{N}_o^s)$$

Since by definition $\zeta^q = \zeta_1^{q_1} \cdot \ldots \cdot \zeta_s^{q_s}$, $\zeta = (\zeta_1,\dots,\zeta_s) \in \mathbb{C}^s$, using the inequality

$$|\zeta_k|^{q_k} \le \frac{|\zeta_k|^{q_k} + |\zeta_k|^{q_k+2}}{\zeta_k^2+1}, \quad k=1,\dots,s,$$

we obtain

$$|\phi^{(q)}(x)| \le \exp(-\langle x,\eta\rangle) \int_{\mathbb{R}^s} (|\zeta|^{q}+|\zeta|^{q+\bar{2}})|\Psi(\zeta)| \cdot$$

$$\cdot \frac{d\xi_1 \ldots d\xi_s}{(\xi_1^2+1)\ldots(\xi_s^2+1)}$$

where $q+\bar{2}$ denotes $(q_1+2,\ldots,q_s+2)$. By assumption we get

$$(18)\quad |\phi^{(q)}(x)| \le \exp(-\langle x,\eta\rangle)\cdot\Big(\frac{B_p}{C_{p",|q|}}\exp(\tilde{m}_{p"}(\eta)) +$$

$$+ \frac{B_{p"}}{C_{p",|q+\bar{2}|}}\exp(\tilde{m}_{p"}(\eta))\Big)$$

where $p" \in \mathbb{N}$, $p" > p$, is chosen so that

$$(19)\qquad \sup\{\frac{C_{p,|q|}}{C_{p",|q+\bar{2}|}}\ ;\ |q| \in \mathbb{N}_o\} < \infty \quad \text{(see C.4).}$$

For $x=(x_1,\ldots,x_s) \in \mathbb{R}^s$, $x_1\cdot\ldots\cdot x_s \neq 0$ we choose each component η_k, $k=1,2,\ldots,s$ of $\eta \in \mathbb{R}^s$ such that $x_k\cdot\eta_k > 0$ for every $k=1,2,\ldots,s$. Taking the infinum by η of the right hand side in (18) we obtain

$$|\phi^{(q)}(x)| \le \bar{B}_{p"}\exp(-m_{p"}(x)\cdot\Big(\frac{1}{C_{p,|q|}} + \frac{1}{C_{p",|q+\bar{2}|}}\Big)$$

for some $\bar{B}_{p"} > 0$, which depends also on the sign of x_k, $k=1,\ldots$ $\ldots,s$. However, we see at once that a constant $\bar{B}_{p"} > 0$ can be found which depends only on $p \in \mathbb{N}$. Hence

$$C_{p,|q|}\cdot\exp(m_p(x))\cdot|\phi^{(q)}(x)| \le \bar{B}_{p"}\Big(\frac{C_{p,|q|}}{C_{p",|q|}} + \frac{C_{p,|q|}}{C_{p",|q+2|}}\Big)$$

and by (19) we get at last

$$\gamma_p(\phi) < \infty .$$

The space of entire analytic functions which satisfy (15) for every $(p,|q|) \in \mathbb{N} \times \mathbb{N}_o$ we denote by $H\{M_p\}$. From Lemmas 1 and 2 we get

THEOREM 9. *The Fourier transformation is a topological isomorphism between* $\sigma\{M_p\}$ *and* $H\{M_p\}$.

The Fourier transform of $T \in \sigma'\{M_p\}$ is an analytic functional $\hat{T}$ on $H\{M_p\}$. We define it in the usual way:

$$<\hat{T},\hat{\phi}> := 2\pi <T(x),\phi(-x)>, \ \phi \in \sigma\{M_p\}.$$

(Obviously $\phi(-x) \in \sigma\{M_p\}$ if $\phi(x) \in \sigma\{M_p\}$).

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REZIME

O KLASI PROSTORA TIPA $S'\{M_p(x, q)\}$

U radu je analizirana struktura prostora $\sigma\{M_p\}$, $\sigma'\{M_p\}$. Pod odredjenim uslovima za matricu $\{C_{p,q}\cdot\exp(M_p(x))\}$ ispitan je odnos prostora $\sigma'\{M_p\}$ i prostora ultradistribucija. Takodje je ispitana Furijerova transformacija na prostorima $\sigma\{M_p\}$, $\sigma'\{M_p\}$.