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NUCLEARITY OF THE SPACE $\sigma\{M_p(x,q)\}$

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ABSTRACT

The aim of this paper is to prove the nuclearity of the space $\sigma\{M_p(x,q)\}$ under suitable conditions on the matrix $\{M_p(x,q)\}$, $p\in N$, $q\in N_o$. This space is investigated in paper [5] (in this journal), so for notations and notions see [5].

1. In the proof of the nuclearity of the space $K\{M_p\}$ in [2] some conditions were supposed. In this part of the paper we are going to discuss some of them. First, we shall repeat some facts from [1].

Let $\{M_p(x)\}$ be a sequence of continuous functions on R such that:

$$0<\delta\leq M_1(x)\leq M_2(x)\leq\dots,\quad x\in R. \tag{1}$$

The space $K\{M_p\}$ is defined as the space of smooth functions ϕ such that

$$\|\phi\|_p:=\sup\{M_p(x)|\phi^{(i)}(x)|;\ \ i\leq p,\ x\in R\}<\infty\ ,\ p\in N\ ;$$

the topology in $K\{M_p\}$ is given by the sequence of norms $\{\|\ \ \|_p\}$.

Moreover, let us suppose that M_p, $p\in N$, monotonically increase as $|x|\to\infty$ (this means if $|x_1|>|x_2|$, $x_1\cdot x_2>0$, then $M_p(x_1)>M_p(x_2)$) and:

(N) For every $p\in N$ there exists $p'\in N$ such that $r_{p,p'}(x):=M_p(x)M_{p'}^{-1}(x)\in L^1(R)$ and $r_{p,p'}(x)\to 0$ monotonically as $|x|\to\infty$

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(I) For every $p \in N$ and every $k \in N_o$, there exist $p' \in N$ and $B_{p,k} > 0$ such that

$$|M_p^{(k)}(x)| \le B_{p,k} M_{p'}(x), \quad x \in R.$$

Under these conditions, it is proved in [2] that $K\{M_p\}$ is nuclear.

The supposition that $M_p(x)$, $p \in N$, are smooth functions and that (I) holds, can be changed in some sense with more general conditions. Namely, the following theorem holds.

THEOREM 1. *Let* $M_p(x)$, $p \in N$, *be a continuous functions on* R *such that* $M_p(x)$ *monotonically increase when* $|x| \to \infty$. *If this sequence satisfies condition* (1) *and*

(T) *There exists* $\varepsilon > 0$ *such that for every* $p \in N$ *there exist* $p' \in N$ *and* $K_{p,p'} > 0$ *such that*

$$M_p(x) \le K_{p,p'} M_{p'}(x-\varepsilon) \quad \text{for} \quad x > K_{p,p'},$$
$$M_p(x) \le K_{p,p'} M_{p'}(x+\varepsilon) \quad \text{for} \quad x < -K_{p,p'},$$

then the space $K\{M_p\}$ *is equal to the space* $K\{N_p\}$ *for a suitable sequence of smooth functions* $\{N_p(x)\}$ *for which* (1) *and* (I) *hold.*

If the sequence $\{M_p\}$ *satisfies condition* (N), *the sequence* $\{N_p\}$ *satisfies this condition as well.*

P r o o f. For the proof we shall use the following construction ([4]).

Let $\omega_1(x) \in C_o^\infty(R)$, supp $\omega_1 \subset [0,\varepsilon]$, $\omega_1(x) \ge 0$ and $\int_R \omega_1(x)dx = 1$.

We define the sequence of smooth functions on the interval $[\varepsilon,\infty)$ by

$$N_p(x) := M_p(x) * \omega_1(x) = \int_0^\varepsilon M_p(x-t)\omega_1(t)dt, \quad x \in [\varepsilon,\infty),\ p \in N.$$

So we have

(2) $$M_p(x-\varepsilon) \le \bar{N}_p(x) \le M_p(x), \quad x \in [\varepsilon,\infty),\ p \in N,$$

(3) $$\bar{N}_p(x) \le \bar{N}_{p+1}(x), \quad x \in [\varepsilon,\infty), \quad p \in N.$$

Similarly, let $\omega_2(x) \in C_0^\infty(R)$, $\mathrm{supp}\,\omega_2 \subset [-\varepsilon,0]$, $\omega_2(x) \geq 0$, $\int_R \omega_2(x)\,dx = 1$, and let

$$\bar{\bar{N}}_p(x) := M_p(x) * \omega_2(x) = \int_{-\varepsilon}^{0} M_p(x-t)\omega_2(t)\,dt, \quad x \in (-\infty,-\varepsilon],\ p \in N.$$

This sequence satisfies the following inequalities .

(2°) $\quad M_p(x+\varepsilon) \leq \bar{\bar{N}}_p(x) \leq M_p(x), \quad x \in (-\infty,-\varepsilon]\ ,\ p \in N\ ,$

(3°) $\quad \bar{\bar{N}}_p(x) \leq \bar{\bar{N}}_{p+1}(x), \quad x \in (-\infty,-\varepsilon]\ ,\ p \in N.$

There exists a sequence of smooth functions $\{N_p(x)\}$ on R such that: $N_p(x)$ is equal to $\bar{N}_p(x)$ on the interval $[\varepsilon,\infty)$; $N_p(x)$ is equal to $\bar{\bar{N}}_p(x)$ on the interval $(-\infty,-\varepsilon]$; $0 < \theta \leq N_p(x) \leq N_{p+1}(x)$ on the interval $(-\varepsilon,\varepsilon)$, $p \in N$.

Thus we construct the sequence of smooth monotonically increasing functions (as $|x| \to \infty$) $\{N_p\}$, which satisfies (1). From (2), (2°) and (T) it follows that for any $p \in N$ there exists $p' \in N$ such that

(2°°) $\quad M_p(x)/K_{p,p'} \leq M_{p'}(x-\varepsilon) \leq N_{p'}(x) \leq M_{p'}(x) \quad$ for $\quad x > K_{p,p'}$,

(3°°) $\quad M_p(x)/K_{p,p'} \leq M_{p'}(x+\varepsilon) \leq N_{p'}(x) \leq M_{p'}(x) \quad$ for $\quad x < -K_{p,p'}$,

Thus the spaces $K\{M_p\}$ and $K\{N_p\}$ are the same in the topological sense.

Condition (I) for the sequence $\{N_p(x)\}$ follows from

$$|N_p^{(k)}(x)| \leq \int_0^\varepsilon M_p(x-t)\,|\omega_1^{(k)}(t)|\,dt \leq M_p(x) \int_0^\varepsilon |\omega_1^{(k)}(t)|\,dt \leq$$

$$\leq C M_p(x) \leq C M_{p'}(x-\varepsilon) \leq C N_{p'}(x)\ , \quad x \geq K_{pp'}\ ,$$

since similar inequalities hold for $x < -K_{p,p'}$.

Let condition (N) holds for the sequence $\{M_p(x)\}$. If $p' \in N$ corresponds to $p \in N$ in (N) let p'' and $K_{p',p''}$ correspond to p' in condition (T). For $x > K_{p',p''}$ we have

$$\frac{N_p(x)}{N_{p''}(x)} \leq \frac{M_p(x)}{M_{p''}(x-\varepsilon)} \leq K_{p',p''} \frac{M_p(x)}{M_{p'}(x)}\ .$$

Since a similar inequality holds for $x<-K_{p',p''}$, it follows that (N) holds for the sequence $\{N_p(x)\}$.

2. In paper [5] we defind the space $\sigma\{M_p(x,q)\}$ by a suitable matrix $\{c_{p,q}\}$ and a suitable sequence of functions $\{\exp(m_p(x))\}$ where we have constructed the sequence $\{m_p(x)\}$ such that this space may be investigated by a Fourier transformation. Since in this paper a Fourier transformation is not needed, we genaralize the conditions for the matrix $\{M_p(x,q)\}$. Namely, we suppose that

$$M_p(x,q) = M_p(x)c_{p,q}, \quad p \in N, \quad q \in N_o,$$

where $\{c_{p,q}\}$ is a matrix which satisfies some of conditions (C.1), (C.2), (C.3), (C.4) from [5] (see remark about (C.1) in [5]):

(C.1) $c_{p,q} \le c_{p+1,q}$ for every $(p,q) \in N \times N_o$;

(C.2) For every $p \in N$ the sequence $\{c_{p,q};\ q \in N_o\}$ monotonically tends to zero when $q \to \infty$;

(C.3) For every $p \in N$ there exists $p' \in N, p' > p$, such that for every $\varepsilon > 0$ there exists $q_o(\varepsilon) \in N$ with the property $c_{p,q} \le \varepsilon\, c_{p',q}$ for $q \ge q_o$;

(C.4) For every $p \in N$ there exists $p' \in N$, such that $\sup \{\frac{c_{p,q}}{c_{p',q+1}};\ q \in N_o\} < \infty$;

and (C.5) (see below), and $\{M_p(x)\}$ is a sequence of functions which satisfies the conditions of Theorem 1. It is clear that the sequence $\{\exp(m_p(x))\}$ from [5] satisfies these conditions.

Let us denote by $E_I^{\{c_{p,q}\}}$, where I is a closed finite interval in R, the space of smooth functions ϕ on I such that

$$\|\phi\|_{p,I} := \sup\{c_{p,q}|\phi^{(q)}(x)|\ ;\ x \in I,\ q \in N_o\} < \infty, \quad p \in N,$$

in which the sequence of norms $\{\|\ \|_{p,I}\}$ defines a topology.

If $\{c_{p,q}\}$, $p \in N$, $q \in N_o$, are of the form

$$c_{p,q} = p^q/N_q, \quad p \in N, \quad q \in N_o,$$

where $\{N_q\}$ is a suitable sequence of positive numbers, then this space becomes the space $\mathcal{E}_I^{(N_p)}$ from [3].

The nuclearity of this space follows from the appropriate condition on $\{N_q\}$ ((M.2)', see [3]). So if we give an appropriate condition on $\{C_{p,q}\}$ we shall obtain the nuclearity of $\mathcal{E}_I(C_{p,q})$.

LEMMA 1. *Let for the matrix* $\{c_{p,q}\}$, (C.1) *and* (C.5) *hold, where:*

(C.5) *For every* $p\in N$ *there exists* $p'\in N$, such that

$$\sum_{q\in N_o} c_{p,q}(c_{p',q})^{-1} < \infty .$$

Then the space $\mathcal{E}_I^{(c_{p,q})}$ *is nuclear.*

The proof of this Lemma is similar to the proof of Proposition 2.4. from [3], so we have omitted it. Let us only remark that (C.3) follows from (C.5).

LEMMA 2. *The sequence of norms* $\{\|\ \|_{p,I}\}$ *on* $\mathcal{E}_I^{(c_{p,q})}$ *is equivalent to the sequence of norms*

$$\|\ \phi\ \|'_{p,I} = \sum_{q=0}^{\infty} c_{p,q} \int_I |\phi^{(q)}(x)|^2 dx)^{1/2},\ p\in N,$$

if (C.1), (C.4) and (C.5) hold.

P r o o f. If $\phi\in \mathcal{E}_I^{(c_{p,q})}$, (C.5) implies

$$\|\phi\|'_{p,I} \le C\ \ \|\phi\|_{p',I} \quad \text{where} \quad C = \int_I dx \sum_{q\in N_o} c_{p,q}(c_{p',q})^{-1}$$

From the Sobolev Lemma (see [6], Theorem 4.1) it follows:

$$\sup\{|\phi(x)|;\ x\in I\} \le \sup\{(\int_I |\phi^{(i)}(x)|^2 dx)^{1/2};\ i=0,1,2\}$$

So we have

$$\sup\{c_{p,q}|\phi^{(q)}(x)|;\ x\in I,\ q\in N_o\} \le \sup\{c_{p'',q}(\int_I |\phi^{(q)}(x)|^2 dx)^{1/2}$$

$$+\ c_{p'',q+1}(\int_I |\phi^{(q+1)}(x)|^2 dx)^{1/2}\ \frac{c_{p,q}}{c_{p'',q+1}} +$$

$$+\ c_{p'',q+2}(\int_I |\phi^{(q+2)}(x)|^2 dx)^{1/2}\ \frac{c_{p,q}}{c_{p'',q+2}} \quad ; \quad q\in N_o\ \}$$

where p" corresponds to p′ in (C.4) and p′ corresponds to p also in (C.4). From this inequality it follows that

$$\|\phi\|_{p,I} \leq (1+2A) \quad \|\phi\|'_{p,I}\ , \quad \text{where}$$

$$A = \sup\{ \frac{c_{p,q}}{c_{p'',q+1}} + \frac{c_{p,q}}{c_{p'',q+2}} \ ; \ q \in N_o \} \ .$$

The constant A exists because (C.4) holds.

Let $\{M_p(x)\}$ satisfies the conditions of Theorem 1 and condition (N); $\{c_{p,q}\}$ satisfies the conditions (C.1), (C.2), (C.3) and (C.4). Let us denote by $\{\gamma_{p,M}\}$ the sequence of norms on $\sigma\{M_p(x,q)\}$ defined by $\gamma_{p,M}=\gamma_p, p\in N$, where $\{\gamma_p\}$ is the sequence of norms in $\sigma\{M_p(x)c_{p,q}\}$ defined by

$$\gamma_p(\phi) := \sup\{c_{p,q}M_p(x)\,|\phi^{(q)}(x)|;\ q\in N_o,\ x\in R\}\ .$$

LEMMA 3. *The sequence of norms* $\{\gamma_{p,M}\}$ *is equivalent with the following two sequences of norms*

(i) $\gamma'_{p,M} := \sup\{c_{p,q}(\int_R |M_p(x)\phi^{(q)}(x)|^2 dx)^{1/2};\ q\in N_o\}, p\in N$

(ii) $\gamma''_{p,N} := \sum_{n=-\infty}^{\infty} \mu_{n,p} \int_n^{n+1} |\phi^{(q)}(x)|^2 dx)^{1/2},\quad p\in N\ ;$

where

$$\mu_{n,p} = N_p(n+1),\quad n\in Z = -N \cup N_o\ .$$

P r o o f. (i) It is clear that every norm from the sequence $\{\gamma'_{p,M}\}$ may be majorized by some norm from the sequence $\{\gamma_{p,M}\}$ (see [2] p. 82).

From (N) it follows that for any $q\in N_o$, $N_p(x)|\phi^{(q)}(x)|\to 0$ as $|x|\to\infty$.

Using this fact and (I), for $x\in R$ we have

$$|N_p(x)\phi^{(q)}(x)| \leq |\int_{-\infty}^{x} (N_p(t)\phi^{(q)}(t))'dt| \leq$$

$$\leq \int_{-\infty}^{\infty} |(N_p(x)\phi^{(q)}(x))'|dx \leq \int_{-\infty}^{\infty} |N'_p(x)\ \phi^{(q)}(x)|dx + \int_{-\infty}^{\infty} |N_p(x)\ \phi^{(q+1)}(x)|dx \leq$$

$$\leq B_{p,1}\int_{-\infty}^{\infty} N_p(x)\,|\phi^{(q)}(x)|dx + \int_{-\infty}^{\infty} N_p(x)\,|\phi^{(q+1)}(x)|dx\ .$$

Multiplying this inequality by $c_{p,q}$,from (C.4) we obtain

(4) $$\gamma_{p,N}(\phi) \leq B_{p,1}\gamma'_{p',N}(\phi) + A\gamma'_{p',N}(\phi)$$

where p' corresponds to p in (C.4) and $A=\sup\{c_{p,q}/c_{p',q+1}\ ,\ q\in N_o\}$.

From Theorem 1, more precisely from (2^{oo}) and (3^{oo}) , it follows that the sequence $\{\gamma_{p,M}\}$on $\sigma\{M_p(x)c_{p,q}\}\equiv\sigma\{N_p(x)c_{p,q}\}$ is equivalent to the sequence $\{\gamma_{p,N}\}$. Using this equivalence for the left hand side of (4), and (2^{oo}) and (3^{oo}) for the right hand side of (4) we obtain that every norm from $\{\gamma_{p,M}\}$ may be majorized by some norm from$\{\gamma'_{p,M}\}$.

(ii) Since the sequences $\{\gamma_{p,M}\}$, $\{\gamma_{p,N}\}$ and $\{\gamma'_{p,N}\}$ are mutually equivalent we ought to prove that $\{\gamma'_{p,N}\}$ and $\{\gamma''_{p,N}\}$ are equivalent. This follows from: condition (I) for $\{N_p(x)\}$; the monotonicity of $N_p(x)$ if $|x|\to\infty$, $p\in N$; condition (T) and

$$\int_R N_p^2(x)|\phi^{(q)}(x)|^2dx = \sum_{n\in Z}\int_n^{n+1} N_p^2(x)|\phi^{(q)}(x)|^2dx .$$

Let us prove this assertion.

If $n\in Z$, $\phi\in\sigma\{N_p(x,q)c_{p,q}\}$ we have

$$\int_n^{n+1} N_p^2(x)|\phi^{(q)}(x)|^2dx \leq \mu_{n,p}^2\int_n^{n+1}|\phi^{(q)}(x)|^2dx ,$$

so $\gamma'_{p,N}(\phi)\leq\gamma''_{p,N}(\phi)$, $p\in N$. $N_p(x)$, $p\in N$ monotonically increase when $|x|\to\infty$, thus for a large enough $|n|$ we have

$$\int_n^{n+1} N_{p_1}^2(x)|\phi^{(q)}(x)|^2dx \geq \mu_{n-1,p_1}^2\int_n^{n+1}|\phi^{(q)}(x)|^2dx \geq$$
$$\geq D_{p,p_1}\mu_{n,p}^2\int_n^{n+1}|\phi^{(q)}(x)|^2dx$$

where we choose p_1 and $D_{p,p_1}>0$ such that

(5) $$N_{p_1}(x-1)\geq D_{p,p_1}N_p(x) \quad \text{for} \quad x>D_{p,p_1} ;$$
$$N_{p_1}(x+1)\geq D_{p,p_1}N_p(x) \quad \text{for} \quad x<-D_{p,p_1} .$$

The existence of p_1 and D_{p,p_1} follows from (T), if this condition is taken m-times where $m\varepsilon \geq 1$. If the numbers p_1 diverge to infinity the corresponding p in (5) also diverge to infinity. So for a suitable C and any $\phi \in \sigma\{M_p(x,q)\}$ we obtain

$$\gamma''_{p_1,N}(\phi) \geq C\, \gamma'_{p,N}(\phi) .$$

Thus the sequences $\{\gamma'_{p,N}\}$ and $\{\gamma''_{p,N}\}$ are equivalent.

3. Now we are ready to prove the following theorem.

THEOREM 2. *Let $\{M_p(x)\}$ be the sequence of functions from Theorem 1 for which* (N) *holds, and let $\{c_{p,q}\}$ be a matrix of positive numbers for which* (C.1) — (C.5) *hold. The corresponding space $\sigma\{M_p(x)c_{p,q}\}$ is nuclear.*

P r o o f. For the proof we need to check that the conditions from the construction of a nuclear space by a known nuclear space, given in [2] p.p. 80, 81, are satisfied.

If we put $m_{n,p} = \mu_{n,p}$, $n \in Z$, $p \in N$, we have

$$m_{n+1,p} \geq m_{n,p} > 0,\ n \in N_o,\ p \in N ,$$

$$m_{n-1,p} \geq m_{n,p} > 0,\ n \in -N,\ p \in N,$$

because $N_p(x)$ increases if $|x|$ increases;

$$m_{n,p} \leq m_{n,p+1}, \text{ because } N_p(x) \leq N_{p+1}(x),\quad p \in N .$$

Let us prove that for any $p \in N$ there exists $p'_2 \in N$ such that

$$\sum_{n\in Z} \frac{m_{n,p}}{m_{n,p'_2}} < \infty . \tag{6}$$

For large enough $n \in N$ we have

$$\frac{m_{n,p}}{m_{n,p_2}} = \frac{N_p(n)}{N_{p_2}(n)} \leq \frac{N_p(n)}{N_{p_2}(n+1)} \leq D_{p,p_2} \frac{N_p(n)}{N_{p_1}(n)}$$

where p_1 corresponds to p in (N) and p_2 is chosen in order to make $N_{p_1}(n) \leq N_{p_2}(n-1)$ hold. The existence of p_2 and D_{p_1,p_2} follows from (T). Namely, if (T) holds for $\{M_p(x)\}$ it is easy to prove that this condition holds for $\{N_p(x)\}$ as well. The convergence of the series follows from (N), By the same arguments we obtain that

$$\sum_{n=-1}^{-\infty} \frac{m_{n,p}}{m_{n,p_2}} < \infty \quad \text{, so (6) holds.}$$

Let us denote by $\Phi(E_n)$ the space of sequences of the form $\phi := =(\phi_0, \phi_1, \phi_{-1}, \phi_2, \phi_{-2} \ldots)$, where $\phi_0 \in E_{[0,1]}^{(c_{p,q})}$, $\phi_1 \in E_{[1,2]}^{(c_{p,q})}$, $\phi_{-1} \in E_{[-1,0]}^{(c_{p,q})}, \ldots,$ such that

$$\|\phi\|_{p,\phi} := \sum_{n \in Z} m_{n,p} \|\phi_n\|_{p,I}^2 < \infty \quad , \ p \in N \ .$$

$(I_n = [n,n+1], \ n \in Z)$.

The proof that $\phi(E_n)$ is nuclear is the same as the proof of of the nuclearity of the space $\phi(M)$ given in [2] p. 81. because the space $E_I^{(c_{p,q})}$ is countable Hilbert nuclear space according to the sequence of scalar products

$$(\phi,\psi)_p = \sum_{q=0}^{\infty} c_{p,q} \int_I \phi^{(q)}(x)\bar{\psi}^{(q)}(x)dx \ , \quad p \in N \ .$$

We embeded the space $\sigma\{N_p(x)c_{p,q}\}$ into the space $\Phi(E_n)$ by the isometry

$$i : \phi \to \{\phi|_{I_n} \ ; \ n = 0,1,-1,\ldots\}$$

The space $i(\sigma\{N_p(x)c_{p,q}\})$ is a closed subspace of $\phi(E_n)$ and so it is nuclear.

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REZIME

NUKLEARNOST PROSTORA $\sigma\{M_p(x, q)\}$

U radu je pokazana nuklearnost prostora $\sigma\{M_p(x,q)\}$ gde je $M_p(x,q) = M_p(x)c_{p,q}$, a $\{M_p(x)\}$ i $\{c_{p,q}\}$ zadovoljavaju odredjene uslove.