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REMARKS ON DIFFERENT SPLITTINGS AND ASSOCIATED GENERALIZED LINEAR METHODS

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ABSTRACT

In this paper we consider some iterative methods for a linear system of equations Ax=b and their connection with the generalized linear method of the Newton-SOR and SOR-Newton type, [12]. Some sufficient conditions for the convergence of the linear method and for the local convergence of the generalized linear method are given.

INTRODUCTION

We shall consider a system of n linear equations with n unknowns, written in a matrix form

$$Ax = b,$$

where A is a nonsingular matrix with nonvanishing diagonal elements. One of the basic principles used in the generation and analysis of the iterative method for linear equations is splitting. That is, for the linear system Ax=b the matrix A is decomposed, or split, into the sum

$$A = B - C$$

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of two matrices, where B is nonsingular and the linear system Bx=d is easy to solve. Then an iterative method is defined by

(1) $x^{m+1} = B^{-1}Cx^m + B^{-1}b, \quad m=0,1,\ldots$.

If we set $H = B^{-1}C$ then (1) can be written as

(2) $x^{m+1} = x^o - (H^m + \ldots + E)B^{-1}(Ax^o - b)$,

where E is the unit matrix. Iteration (2) is convergent for all x^o if and only if the spectral radius $\rho(H)$ of matrix H is less than 1.

We shall now give some splittings of A and associated iterations of form (2). Let

$$A = D - T - S$$

be the decomposition of A into diagonal, strictly lower triangular, and strictly upper triangular parts. Let $F = \mathrm{diag}(f_1,\ldots,f_n)$ be a nonsingular matrix. Let for $\omega,\sigma \in \mathbb{R}$, $\omega \neq 0$,

$$B = \omega^{-1}(F-\sigma T), \quad C = \omega^{-1}(F-\omega D+(\omega-\sigma)T + \omega S) .$$

Matrix B is nonsingular, $A = B-C$ and

(3) $B^{-1}C = (F-\sigma T)^{-1}(F-\omega D + (\omega-\sigma)T + \omega S)$.

We denote by $H(F,\omega,\sigma)$ the matrix $B^{-1}C$ and by VAOR the associated iteration (2). If $F = D$ the VAOR reduces to AOR, [6] . Further, the AOR method for specific values of the parameters ω,σ reduces to well-known methods: for $\sigma = 0$, $\omega = 1$ the AOR method is the Jacobi method, for $\sigma = \omega = 1$ the AOR method is the Gauss-Seidel method, for $\sigma = 0$ the AOR method is the JOR method, for $\sigma = \omega$ the AOR method is the SOR method.

The sufficient conditions for the convergence of the AOR methods are considered by many authors including [1], [2], [6] ,[7] ,[9] ,[10] ,[11] . In this paper we shall give some sufficient conditions for the convergence of the VAOR method. Using these results we can also give some sufficient conditions for the local convergence of some generalized linear methods for the numerical solution of the system of nonlinear equations.

Our results include some results from [5] and [12].

ON CONVERGENCE OF THE VAOR METHOD

We shall begin with some notations:

For $A=[a_{ij}]\in C^{n,n}$ (= set of complex $n\times n$ matrices) we define for $i=1,2,\dots,n$

$$P_i(A)=\sum_{j\in N(i)}|a_{ij}|,\quad Q_i(A)=\sum_{j\in N(i)}|a_{ji}|,\quad T_i=\max_{j\in N(i)}|a_{ji}|,$$

where $N=\{1,2,\dots,n\}$, $N(i)=N\setminus\{i\}$.

THEOREM 1. *Let* $A=[a_{ij}]\in C^{n,n}$ *be such that*

(4) $$|a_{ii}||a_{jj}|>P_i(A)P_j(A),\quad i\in N,\quad j\in N(i),$$

or

(5) $$\alpha\in[0,1],\quad |a_{ii}|>P_i^{\alpha}(A)Q_i^{1-\alpha}(A),\quad i\in N,$$

or

(6) $$\alpha\in[0,1],\quad |a_{ii}|>\alpha P_i(A)+(1-\alpha)Q_i(A),\quad i\in N.$$

Let $F=\operatorname{diag}(f_1,\dots,f_n)\in C^{n,n}$, *and let* $f_i/a_{ii}\in\mathbb{R}$, $f_i/a_{ii}>0$, $i\in N$, $q=\min_{i\in N}\frac{f_i}{a_{ii}}$. *Then, for* $\omega\in(0,q]$, $\sigma\in[0,q]$, *we have* $\rho(H(F,\omega,\sigma))<1$.

P r o o f. The iteration matrix $H(F,\omega,\sigma)$ of the VAOR method is defined by (3). We assume that for some eigenvalue λ of $H(F,\omega,\sigma)$ $|\lambda|\geq 1$ holds. For this eigenvalue we have the following relation

$$\det(H(F,\omega,\sigma)-\lambda E)=0.$$

Since $\det(F-\sigma T)=\det(F)\neq 0$, this is equivalent to $\det(Q)=0$, where $Q=[q_{ij}]$ is defined by

$$Q=(F-\sigma T)(H(F,\omega,\sigma)-\lambda E)=(1-\lambda)F-\omega D+(\omega+\sigma(\lambda-1))T+\omega S.$$

In [6] it is proved that for $|\lambda| \geq 1$, $0 < x \leq 1$, $0 \leq y \leq 1$ we have

$$|\lambda-1+x| \geq |y(\lambda-1)+x|, \quad |\lambda-1+x| \geq x .$$

Now it follows that

$$|q_{ii}| = |(\lambda-1)f_i+\omega a_{ii}| = |\lambda-1+\omega \frac{a_{ii}}{f_i}|\,|f_i| \geq \omega|a_{ii}| ,$$

and

$$|q_{ii}| \geq |(\lambda-1)\frac{\sigma a_{ii}}{f_i}+\omega \frac{a_{ii}}{f_i}|\,|f_i| = |(\lambda-1)\sigma+\omega|\,|a_{ii}| ,$$

since $0 < \omega a_{ii}/f_i \leq 1$, $0 \leq \sigma a_{ii}/f_i \leq 1$. It is easy to show that

$$P_i(Q) \leq |\frac{q_{ii}}{a_{ii}}|P_i(A) , \quad Q_i(Q) \leq |\frac{q_{ii}}{a_{ii}}|Q_i(A), \quad i \in N ,$$

and

$$P_i(Q)P_j(Q) < |q_{ii}||q_{jj}|, \quad i \in N,\ j \in N(i) ,$$

if (4) is true,

$$P_i^{\alpha}(Q)Q_i^{1-\alpha}(Q) < |q_{ii}| , \quad i \in N ,$$

if (5) is true,

$$\alpha P_i(Q) + (1-\alpha)Q_i(Q) < |q_{ii}| , \quad i \in N$$

if (6) is true.

It follows now from [8], 2.4.1, 2.5.1, 2.5.2, that $\det Q \neq 0$. This contradicts the singularity of $H(F,\omega,\sigma)-\lambda E$. Therefore, $\rho(H(F,\omega,\sigma) < 1$.

COROLLARY. *Let* $A = [a_{ij}] \in C^{n,n}$ *be either strictly or irreducibly diagonally dominant. Let* $F = \mathrm{diag}(f_1, \ldots, f_n) \in C^{n,n}$ *be nonsingular and let*

$$f_i/a_{ii} \in \mathbb{R}, \quad f_i/a_{ii} > 0, \quad i \in N, \quad q = \min_{i \in N} \frac{f_i}{a_{ii}} .$$

Then $\rho(H(F,\omega,\sigma)) < 1$ *for* $\omega \in (0,q]$, $\sigma \in [0,q]$.

P r o o f. We see that the eigenvalues of $H(F,\omega,\sigma)$ are the roots of $\det Q = 0$, with Q given in Theorem 1. With $|\lambda| \geq 1$, we know that Q is irreducible when A is irreducible. If A is (strictly) diagonally dominant, then Q is also a (strictly) diagonally dominant matrix. With these conditions, the value of λ such that $|\lambda| \geq 1$ can not be the eigenvalue of $H(F,\omega,\sigma)$ because Q is nonsingular.

THEOREM 2. *Let* $A = [a_{ij}]$, $F = \mathrm{diag}(f_1,\dots,f_n) \in C^{n,n}$ *with* $a_{ii}, f_i > 0$, $i \in N$, *and let* $q = \min_{i\in N} \frac{f_i}{a_{ii}}$. *Then* $\rho(H(F,\omega,\sigma)) < 1$ *for* $\omega \in (0,q]$, $\sigma \in [0,q]$ *if*

$$a_{ii} > \min(P_i(A), T_i(A)), \quad i \in N,$$

$$a_{ii} + a_{jj} > P_i(A) + P_j(A), \quad i \in N,\ j \in N(i).$$

P r o o f. As in the proof of Theorem 1 the eigenvalues of the matrix $H(F,\omega,\sigma)$ are the roots of $\det Q = 0$. With $|\lambda| \geq 1$ we have

$$T_i(Q) \leq \frac{|q_{ii}|}{a_{ii}} T_i(A), \quad P_i(Q) \leq \frac{|q_{ii}|}{a_{ii}} P_i(A), \quad i \in N,$$

and

$$\min(P_i(Q), T_i(Q)) \leq \frac{|q_{ii}|}{a_{ii}} \min(P_i(A), Q_i(A)) < |q_{ii}|. \tag{7}$$

Further,

$$|q_{ii}| + |q_{jj}| \geq |q_{ii} + q_{jj}| = |(\lambda-1)(f_i+f_j) + \omega(a_{ii}+a_{jj})|.$$

Since,

$$\frac{f_i + f_j}{a_{ii}+a_{jj}} \geq \min\left(\frac{f_i}{a_{ii}}, \frac{f_j}{a_{jj}}\right) \geq q, \quad i \in N,\ j \in N(i),$$

we have, as in the proof of Theorem 1,

$$|q_{ii}+q_{jj}| \geq \omega(a_{ii}+a_{jj}),$$
$$|q_{ii}+q_{jj}| \geq |(\lambda-1)\sigma + \omega|(a_{ii}+a_{jj}).$$

Now it holds for $i \in N$, $j \in N(i)$

$$(8) \qquad P_i(Q) + P_j(Q) \le |q_{ii}+q_{jj}| \frac{P_i(A)+P_j(A)}{a_{ii}+a_{jj}} < |q_{ii}|+|q_{jj}| .$$

From (7) and (8) it follows by Theorem 5 from [13] that $\det Q = 0$, which contradicts the singularity of $H(F,\omega,\sigma)-\lambda E$.

ON SOME GENERALIZED LINEAR METHODS

In this section we shall consider the system of nonlinear equations $Gx = 0$, where $G: D \subset \mathbb{R}^n \to \mathbb{R}^n$, and suppose that G is F-differentiable and G' is continuous on an open neighbourhood $S_o \subset D$ of a point x^* for which $Gx^* = 0$.

One way to utilize the VAOR iteration in connection with nonlinear equations is to get approximate solutions of the linear systems which must be solved to carry out Newton's method. In this case, we would have a composite Newton-VAOR iteration, with Newton's method as the primary iteration and VAOR as the secondary iteration. In [12] it is shown that such a combination can be written in the form

$$(9) \qquad x^{k+1} = x^k - (E+\ldots+H(x^k)^{m-1})B(x^k)^{-1}Gx^k, \quad k=0,1,\ldots, \quad m \ge 1 ,$$

where B and H are defined by

$$B(x) = \omega^{-1}(F(x) - \sigma T(x)) ,$$

$$C(x) = \omega^{-1}(F(x) - \omega D(x) + (\omega-\sigma)T(x) + \omega S(x)) ,$$

$$H(x) = B(x)^{-1}C(x), \quad \omega,\sigma \in \mathbb{R} , \quad \omega \ne 0 .$$

In this case $F(x)$ is any nonsingular matrix and

$$G'(x) = D(x) - T(x) - S(x)$$

is the decomposition of $G'(x)$ into its diagonal, strictly lower, and strictly upper triangular parts, and it is assumed that $D(x)$ is nonsingular.

Under the above assumptions, x^* is a point of attraction of the iteration defined by (9) if $B: S_o \to \mathbb{R}^{n,n}$ is continuous at x^*, $B(x^*)$ nonsingular and if $\rho(H(x^*)) < 1$, 10.3.1 from [12].

We can now use the results of the previous sections to obtain some sufficient conditions for a local convergence of the Newton-VAOR method, if we apply Theorem 1 and 2 to $H(x^*)$.

We have considered linear iterative methods in their traditional role of solving linear systems. However, it is also possible to give a direct extension of these methods to nonlinear equations, [12]. So, we have the one step SOR-Newton process and some of its modifications and generalizations [3], [4], [5], [12]. The one-step vSOR-Newton method, [5], is given by

$$x_i^{k+1} = x_i^k - \omega \frac{g_i(x^{k,i})}{f_i(x^{k,i})}, \quad i=1,\dots,n,$$

where, as usual, $g_1,\dots,g_n$ are the components of $G: D \subset \mathbb{R}^n \to \mathbb{R}^n$,

$$\omega \in \mathbb{R}\setminus\{0\}, \quad x^{k,i} = [x_1^{k+1},\dots,x_{i-1}^{k+1}, x_i^k,\dots,x_n^k]^T, \quad i=1,\dots,n,$$

and functions $f_i: D \to \mathbb{R}$, $i=1,\dots,n$ are continuous on D with

$$f_i(x) > 0, \quad x \in D, \quad i=1,\dots,n.$$

For $f_i(x) = g_i'(x)$, $i=1,\dots,n$, this method reduces to the one-step SOR-Newton method.

In [3] a generalization of the vSOR-Newton method is given. This is the one-step vAOR-Newton method

$$x_i^{k+1} = x_i^k - \omega \frac{g_i(z^k)}{f_i(z^k)}, \quad i=1,\dots,n,$$

(10)
$$z_1^k = x_1^k - \frac{g_1(x^k)}{f_1(x^k)}, \quad z_i^k = x_i^k - \sigma \frac{g_i(z^{k,i})}{f_i(z^{k,i})}, \quad i=2,\dots,n,$$

where

$$\omega,\sigma \in \mathbb{R}\setminus\{0\}, \quad z^{k,i} = [z_1^k,\dots,z_{i-1}^k, x_i^k,\dots,x_n^k]^T, \quad i=1,\dots,n,$$

and $f_i: D \to \mathbb{R}$ are continuous on D and $f_i(x) > 0$ for $x \in D$, $i=1,\dots,n$.

This method reduces to the vSOR-Newton method if $\sigma=\omega$.

Clearly, (10) may be written in the form $x^{k+1}=G_{\omega,\sigma}x^k$ although now the mapping $G_{\omega,\sigma}$ becomes rather complicated. In [3] it is shown that

$$G'_{\omega,\sigma}(x)=(F(x)-\sigma T(x))^{-1}(F(x)-\omega D(x)+(\omega-\sigma)T(x)+\omega S(x)) .$$

Now it is easy to see that $G'_{\omega,\sigma}(x)=H(x)$ with $F(x)=\mathrm{diag}(f_1(x),\ldots,f_n(x))$.

To prove the local convergence of the vAOR-Newton method it sufficies to show that $G_{\omega,\sigma}$ is F-differentiable at x^* and that $\rho(G'_{\omega,\sigma}(x^*))<1$, see the Ostrowski theorem, [12] . So, for the local convergence of the Newton-VAOR and vAOR-Newton method we consider the same matrix $H(x^*)$. We may apply the results of the previous section to $H(x^*)$ in order to obtain some sufficient conditions for the local convergence of these methods. In [3], [4] are given some sufficient conditions for the local convergence of the vAOR-Newton method. As special cases of this method we have the vSOR-Newton method from [5], and the SOR-Newton method from [12]. The Newton-VAOR method also contains, as a special case, the m-step Newton-SOR method $(m\geq 1)$. Now, we can summarize our consideration of the system of nonlinear equations in the next theorem.

THEOREM 3. *Let* $G:D\subset\mathbb{R}^n\to\mathbb{R}^n$ *be F-differentiable in an open neighbourhood* $S_0\subset D$ *of a point* $x^*\in D$ *at which* G' *is continuous and* $Gx^*=0$. *Let* $D(x)$, $-T(x)$, $-S(x)$ *be the diagonal strictly lower, and strictly upper triangular parts of* $G'(x)$. *Suppose that* $f_i(x)>0$, $i=1,\ldots,n$, *are continuous on* D *and* $F=\mathrm{diag}(f_1(x),\ldots,f_n(x))$. *If* $G'(x^*)=[g_{ij}]$ *and*

$$g_{ii}>0,\ g_{ii}g_{jj}>P_i(G'(x^*))P_j(G'(x^*)),\ i\in N,\ j\in N(i)\ ,$$

or

(11) $\alpha\in[0,1]$, $g_{ii}>P_i^{\alpha}(G'(x^*))Q_i^{1-\alpha}(G'(x^*))$, $i\in N$,

or

$$\alpha \in [0,1], \quad g_{ii} > \alpha P_i(G'(x^*)) + (1-\alpha)Q_i(G'(x^*)), \quad i \in N,$$

or

$$g_{ii} > \min(P_i(G'(x^*)), T_i(G'(x^*)), \quad i \in N,$$

$$g_{ii} + g_{jj} > P_i(G'(x^*)) + P_j(G'(x^*)), \quad i \in N, \; j \in N(i),$$

or

$G'(x^*)$ *is irreducibly diagonally dominant,*

$G'(x^*)$ *is an M-matrix,*

then x^* *is a point of attraction of the Newton-VAOR and vAOR-Newton iteration for* $\sigma, \omega \in (0,q]$, *where* $q = \min_{i \in N} f_i(x^*)/g_{ii}$.

The proof of this theorem in the case that $G'(x^*)$ is an M-matrix, $0 < \sigma \leq \omega \leq q$ is given in [3]. If $G'(x^*)$ is strictly diagonally dominant, then (11) is true for $\alpha = 1$, and it follows that the statement of Theorem 3 also holds in this case.

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REZIME

PRIMEDBE NA RAZLIČITE DEKOMPOZICIJE I
PRIDRUŽENE UOPŠTENE LINEARNE METODE

U radu se posmatraju neki iterativni postupci za rešavanje sistema linearnih jednačina Ax=b nastali dekomponovanjem matrice A u sumu A=B-C dve matrice, gde je B nesingularna matrica i takva da se sistem Bx=d može "lako rešiti". Formiran je VAOR iterativni postupak za iterativno rešavanje sistema Ax=b, koji kao specijalne slučajeve sadrži AOR, SOR i JOR postupke. Dati su neki dovoljni uslovi za konvergenciju VAOR postupka. Takodje se posmatraju kombinacije nelinearno-linearnih i linearno-nelinearnih postupaka za iterativno rešavanje sistema nelinearnih jednačina. Dati su neki dovoljni uslovi za lokalnu konvergenciju ovih postupaka. Kao specijalni slučajevi posmatranih postupaka javljaju se postupci Newton-SOR, SOR-Newton, [12], i vSOR-Newton, [5].