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ON A LOCAL CONVERGENCE OF THE vAORN METHOD

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ABSTRACT

In this paper we consider a method for the numerical solution of nonlinear systems of equations. The method is a two-parameter generalization of the vSOR-Newton method (vSORN). When the two parameters involved are equal, it coincides with the vSORN method from [1] as a special case. This method we call vAORN ("verallgemeinerte" Accelerated Overrelaxation Newton) method.

1. INTRODUCTION

We shall consider the system of nonlinear equations

$$Fx = 0,$$

where $F: S \subset \mathbb{R}^n \to \mathbb{R}^n$.

For some $x^0 \in S$ and some $\sigma, \omega \in \mathbb{R}$, $\omega \neq 0$, the iterates $\{x^k\}$ are defined by

$$\text{(vAORN)} \qquad x_i^{k+1} = x_i^k - \omega \frac{F_i(z^k)}{d_i(z^k)}, \quad i=1,2,\dots,n;\ k=0,1,\dots,$$

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where $z_1^k = x_1^k - \sigma \frac{F_1(x^k)}{d_1(x^k)}$, $z_i^k = x_i^k - \sigma \frac{F_i(z^{k,i})}{d_i(z^{k,i})}$, $i=2,3,\dots,n$,

$$z^{k,i} = [z_1^k,\dots,z_{i-1}^k, x_i^k,\dots,x_n^k]^T ,$$

and $d_i : S \to \mathbb{R}$, $i=1,2,\dots,n$.

We assume that:

1) F is F-differentiable on an open neighborhood $S_o \subset S$ of a point x^*, for which $Fx^* = 0$.

2) The functions d_i, $i=1,2,\dots,n$ are continuous on S and $d_i(x) > 0$, $i=1,2,\dots,n$, $x \in S$.

3) $f_i = \frac{\partial F_i}{\partial x_i}(x^*) \neq 0$, $i=1,2,\dots,n$, and without any restriction of the generality we can suppose that $f_i > 0$, $i=1,2,\dots,n$. Under these assumptions we shall prove the local convergence of the vAORN method using the theorem of Ostrowski, [3].

In case that $\sigma=\omega$ the vAORN method reduces to the vSORN method from [1]. In this case, if $F'(x^*)$ is a strictly diagonally dominant matrix, we get the convergence interval I_o for ω wider than the one from [1]. For $\sigma \in I_o$, using Theorem 1 from [4], we get a narrower convergence interval for ω than in [4]. In case that $d_i(x) = \frac{\partial F_i}{\partial x_i}(x)$, $i=1,2,\dots,n$ and $Fx = Ax + b$, where $A \in \mathbb{R}^{n,n}$ (= set of real $n \times n$ matrices) and $b \in \mathbb{R}^n$, the vAORN method is the AOR method from [2].

Let $G_{\sigma,\omega}$ be an iteration function for (vAORN) and let $F'(x^*) = D_F - L_F - U_F$ be the decomposition of $F'(x^*)$ into its diagonal, strictly lower, and strictly upper triangular parts. Let $D = \mathrm{diag}(d_1(x^*), d_2(x^*),\dots,d_n(x^*))$. For $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and $\alpha \in [0,1]$ we define for $i=1,2,\dots,n$

$$P_i(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| , \quad Q_i(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ji}| ,$$

$$P_{i,\alpha}(A) = \alpha P_i(A) + (1-\alpha) Q_i(A) .$$

2. THE LOCAL CONVERGENCE OF THE vAORN METHOD

Let $\sigma \neq 0$ and let G_σ be an iteration function for the vSORN method

$$\text{(vSORN)} \qquad x_i^{k+1} = x_i^k - \sigma \frac{F_i(x^{k,i})}{d_i(x^{k,i})}, \quad i=1,2,\dots,n; \quad k=0,1,\dots$$

with $\qquad x^{k,i} = [x_1^{k+1},\dots,x_{i-1}^{k+1}, x_i^k, \dots, x_n^k]^T$,

from [1]. Then $G_{\sigma,\omega} = (1-\frac{\omega}{\sigma}) E + \frac{\omega}{\sigma} G_\sigma$,

Since G_σ is F-differentiable at x^* (Theorem 1 from [1]), $G_{\sigma,\omega}$ is also F-differentiable at the same point and

$$(1) \qquad G'_{\sigma,\omega}(x^*) = (D-\sigma L_F)^{-1}(D-\omega D_F + (\omega-\sigma)L_F + \omega U_F) .$$

For $\sigma = 0$, $G_{0,\omega}(x) = x-\omega D^{-1}(x)F(x)$ and $G'_{0,\omega}(x^*) = E - \omega D^{-1}F'(x^*)$, which is a special case of (1) for $\sigma = 0$. Thus, (1) is true for $\sigma,\omega \in \mathbb{R}$, $\omega \neq 0$.

From now on we shall assume that the assumptions 1)-3) from the introduction are valid.

THEOREM 1. *Let* $\alpha \in [0,1]$ *and let* $d_i - |\sigma| P_{i,\alpha}(L_F) > 0$, $i=1,2,\dots,n$. *Then*

$$\rho(G'_{\sigma,\omega}(x^*)) \leq \max_i \frac{|d_i-\omega f_i| + |\omega-\sigma| P_{i,\alpha}(L_F) + |\omega| P_{i,\alpha}(U_F)}{d_i - |\sigma| P_{i,\alpha}(L_F)} .$$

P r o o f. Let λ be any eigenvalue of $G'_{\sigma,\omega}(x^*)$ and suppose that

$$|\lambda| > \frac{|d_i-\omega f_i| + |\omega-\sigma| P_{i,\alpha}(L_F) + |\omega| P_{i,\alpha}(U_F)}{d_i - |\sigma| P_{i,\alpha}(L_F)}, \quad i=1,2,\dots,n .$$

After some manipulations we have

$$|a_{ii}| = |(\lambda-1)d_i+\omega f_i| \geq \alpha(|\omega+\sigma(\lambda-1)|P_i(L_F) + |\omega|P_i(U_F)) +$$
$$+ (1-\alpha)(|\omega+\sigma(\lambda-1)|Q_i(L_F)+|\omega|Q_i(U_F)) =$$
$$=\alpha P_i(A) + (1-\alpha)Q_i(A), \quad i=1,2,\ldots,n ,$$

where $A=[a_{ij}] \in \mathbb{R}^{n,n}$, $A=(\lambda-1)D+\omega D_F-(\omega+\sigma(\lambda-1))L_F-\omega U_F$. Then Theorem 2.5.2. from [6] shows that $\det A \neq 0$. Since $(D-\sigma L_F)(\lambda E - G'_{\sigma,\omega}(x^*)) = A$ and $\det(D-\sigma L_F) \neq 0$, it follows $\det(\lambda E-G'_{\sigma,\omega}(x^*)) \neq 0$. This contradicts the singularity of $\lambda E - G'_{\sigma,\omega}(x^*)$.

THEOREM 2. *Let for some* $\alpha \in [0,1]$, $f_i > P_{i,\alpha}(F'(x^*))$, $i=1,2,\ldots,n$. *Then for*

$$0 < \omega < \min_i \frac{2d_i}{f_i+P_{i,\alpha}(F'(x^*))}$$

and

$$\max_i \frac{-\omega(f_i-P_{i,\alpha}(F'(x^*)))+2\max(0,\omega f_i-d_i)}{2P_{i,\alpha}(L_F)} <$$
$$< \sigma < \min_i \frac{\omega(f_i+P_{i,\alpha}(L_F)-P_{i,\alpha}(U_F))+2\min(0,d_i-\omega f_i)}{2P_{i,\alpha}(L_F)}$$

$\rho(G'_{\sigma,\omega}(x^*)) < 1$ *holds, i.e. the vAORN method converges locally.*

P r o o f. We shall prove that for all $i=1,2,\ldots,n$, the following implication holds.

$$\text{(2)} \quad \left.\begin{array}{l} 0 < \omega < \dfrac{2d_i}{f_i+P_{i,\alpha}(F'(x^*))} \\[2ex] \dfrac{-\omega(f_i-P_{i,\alpha}(F'(x^*)))+2\max(0,\omega f_i-d_i)}{2P_{i,\alpha}(L_F)} < \sigma < \\[2ex] < \dfrac{\omega(f_i+P_{i,\alpha}(L_F)-P_{i,\alpha}(U_F))+2\min(0,d_i-\omega f_i)}{2P_{i,\alpha}(L_F)} \end{array}\right\} \Rightarrow$$

$$\text{(3)} \quad \frac{|d_i-\omega f_i|+|\omega-\sigma|P_{i,\alpha}(L_F)+|\omega|P_{i,\alpha}(U_F)}{d_i-|\sigma|P_{i,\alpha}(L_F)} < 1 .$$

Since for σ and ω from (2) we have $d_i-|\sigma|P_{i,\alpha}(L_F) > 0$, Theorem 1 and (3) show that $\rho(G'_{\sigma,\omega}(x^*)) < 1$.

Let us introduce the following notations: $\ell_i = P_{i,\alpha}(L_F)$, $u_i = P_{i,\alpha}(U_F)$.

To prove implications (2) => (3) we consider the next cases.

Case I: $0 < \omega \le \dfrac{d_i}{f_i}$, $\dfrac{-\omega(f_i-\ell_i-u_i)}{2\ell_i} < \sigma \le 0$.

Then $d_i-\omega f_i+\omega\ell_i-\sigma\ell_i+\omega u_i < d_i+\sigma\ell_i$.

Case II: $0 < \omega \le \dfrac{d_i}{f_i}$, $0 < \sigma \le \omega$.

Then $d_i-\omega f_i+\omega\ell_i-\sigma\ell_i+\omega u_i < d_i-\sigma\ell_i$, since $\ell_i+u_i < f_i$.

Case III: $0 < \omega \le \dfrac{d_i}{f_i}$, $\omega < \sigma < \dfrac{\omega(f_i+\ell_i-u_i)}{2\ell_i}$.

Then $d_i-\omega f_i+\sigma\ell_i-\omega\ell_i+\omega u_i < d_i-\sigma\ell_i$.

Case IV: $\dfrac{d_i}{f_i} < \omega < \dfrac{2d_i}{f_i+\ell_i+u_i}$, $\dfrac{\omega(f_i+\ell_i+u_i)-2d_i}{2\ell_i} < \sigma \le 0$.

Then $\omega f_i-d_i+\omega\ell_i-\sigma\ell_i+\omega u_i < d_i+\sigma\ell_i$.

Case V: $\dfrac{d_i}{f_i} < \omega < \dfrac{2d_i}{f_i+\ell_i+u_i}$, $0 < \sigma \le \omega$.

Then $\omega f_i-d_i+\omega\ell_i-\sigma\ell_i+\omega u_i < d_i-\sigma\ell_i$.

Case VI: $\dfrac{d_i}{f_i} < \omega < \dfrac{2d_i}{f_i+\ell_i+u_i}$, $\omega < \sigma < \dfrac{\omega(-f_i+\ell_i-u_i)+2d_i}{2\ell_i}$.

Then $\omega f_i-d_i+\sigma\ell_i-\omega\ell_i+\omega u_i < d_i-\sigma\ell_i$.

COROLLARY 2.1. *Let $F'(x^*)$ be a strictly diagonally dominant matrix. Then for*

$$0 < \omega < \min_i \frac{2d_i}{f_i+P_i(F'(x^*))} \quad \textit{and}$$

$$\max_i \frac{-\omega(f_i - P_i(F'(x^*))) + 2\max(0, \omega f_i - d_i)}{2P_i(L_F)} < \sigma <$$

$$< \min_i \frac{\omega(f_i + P_i(L_F) - P_i(U_F)) + 2\min(0, d_i - \omega f_i)}{2P_i(L_F)}$$

$\rho(G'_{\sigma,\omega}(x^*)) < 1$ *holds, i.e. the vAORN method converges locally.*

The proof follows immediately from Theorem 2 with $\alpha = 1$.

COROLLARY 2.2. *Let $F'(x^*)$ be a strictly diagonally dominant matrix. For the iteration function G_ω of the vSORN method the following implication holds*

$$0 < \omega < \min_i \frac{2d_i}{f_i + P_i(F'(x^*))} \Rightarrow \rho(G'_\omega(x^*)) < 1 .$$

P r o o f. For $\omega = \sigma$ we have $G'_{\omega,\omega}(x^*) = G'_\omega(x^*)$. Since for any $i = 1, 2, \ldots, n$, $-\omega(f_i - P_i(F'(x^*))) + 2\max(0, \omega f_i - d_i) < 0$ and

$$\omega < \frac{2d_i}{f_i + P_i(F'(x^*))} \Rightarrow \omega < \frac{\omega(f_i + P_i(L_F) - P_i(U_F)) + 2\min(0, d_i - \omega f_i)}{2P_i(L_F)},$$

using Theorem 2 we complete the proof.

REMARK 1. The local convergence of the vSORN method was proved in [1] for $\omega \in (0, q]$, where $q = \min_i \frac{d_i}{f_i}$ under the same assumptions as in Corollary 2.2. Our interval for ω is wider.

REMARK 2. Theorem 2 enables us to consider the local convergence of the vSORN method for a wider class of matrices than in [1].

REMARK 3. From Corollary 2.2. and Theorem 1 from [4] it follows that

$$0 < \omega \leq \sigma < \min_i \frac{2d_i}{f_i + P_i(F'(x^*))} \Rightarrow \rho(G'_{\sigma,\omega}(x^*)) < 1 .$$

Our Theorem 2 gives us more.

REMARK 4. For $d_i = \frac{\partial F_i}{\partial x_i}$, $Fx = Ax + b$, $A \in \mathbb{R}^{n,n}$, $b \in \mathbb{R}^n$, Theorem 2 is a special case of Theorem 3 from [5].

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REZIME

O LOKALNOJ KONVERGENCIJI VAORN POSTUPKA

U radu se posmatra postupak za numeričko rešavanje sistema nelinearnih jednačina Fx=0. Taj postupak, koji je dvoparametarska generalizacija vSOR-Njutnovog postupka (vSORN), razmatranog u [1], nazvali smo vAORN ("verállgemeinerte" Accelerated Overrelaxation Newton) postupak. Pod odredjenim pretpostavkama za funkciju F i matricu $F'(x^*)$, gde je x^* rešenje sistema Fx=0, odredjeni su intervali konvergencije za parametre σ i ω. U specijalnom slučaju, za $\sigma=\omega$ i kada je $F'(x^*)$ strogo dijagonalno dominantna matrica, interval konvergencije za ω dobijen u ovom radu širi je od odgovarajućeg iz [1]. Analogni rezultati iz ovog rada za slučaj sistema linearnih jednačina dati su u [5].