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A NOTE ON PRODUCT CURVATURE TENSORS

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ABSTRACT

The purpose of the present note is to find some relations connecting the product conformal curvature tensor, the product projective curvature tensor, the product concircular curvature tensor and the product conharmonic curvature tensor.

1. An n-dimensional differentiable manifold M_n of class C^∞ is called a locally decomposable Riemannian space [1] if in M_n a linear transformation field $F \neq I$ and a positive definite Riemannian metric g are given, satisfying

$$F^2 = I, \quad g(X,Y) = g(FX,FY), \quad (\nabla_X F)(Y) = 0$$

for any vector fields X and Y on M_n, where I denotes the identity transformation field and ∇ is the operator of the covariant derivative with respect to the Riemannian metric g. Putting

$$F(X,Y) = g(FX,Y) = g(X,FY)$$

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we have

$$F(X,Y) = F(Y,X).$$

The matrix (F_j^i) has ± 1 as the proper values. Let us denote by $T(x)$ the tangent vector space of M_n at a point P and let $P(x)$ and $Q(x)$ be the proper vector spaces corresponding to the proper values $+1$ and -1 respectively. If we put $\dim P(x) = p$, $\dim Q(x) = q$, then p and q are constants and it holds that $\phi = F_i^i = p-q$. In the following we suppose $p > 2$, $q > 2$.

Considering the infinitesimal product conformal and infinitesimal product projective transformations on a locally decomposable Riemannian space, S.Tachibana [2] obtained the product conformal curvature tensor.

$$\begin{aligned} C(X,Y,Z) &= K(X,Y,Z) \\ &+ a[K(X,Y)Z - g(X,Z)K(Y) + K(X,FY)FZ - g(X,FZ)K(FY) \\ &- K(X,Z)Y + g(X,Y)K(Z) - K(X,FZ)FY + g(X,FY)K(FZ)] \\ &- b[K(X,FY)Z - g(X,FZ)K(Y) + K(X,Y)FZ - g(X,Z)K(FY) \\ &- K(X,FZ)Y + g(X,FY)K(Z) - K(X,Z)FY + g(X,Y)K(FZ)] \\ &+ [K(a\alpha_1-b\beta_1)+K^*(a\beta_1-b\alpha_1)][g(X,Z)Y-g(X,Y)Z \\ &+ g(X,FZ)FY - g(X,FY)FZ] + [K(a\beta_1-b\alpha_1) + \\ &+ K^*(a\alpha_1-b\beta_1)][g(X,FZ)Y - g(X,FY)Z + g(X,Z)FY - g(X,Y)FZ] \end{aligned} \tag{1}$$

and product projective curvature tensor

$$\begin{aligned} P(X,Y,Z) &= K(X,Y,Z) \\ &+ \alpha_1[K(X,Z)Y - K(X,Y)Z + K(X,FZ)FY - K(X,FY)FZ] \\ &+ \beta_1[K(X,FZ)Y - K(X,FY)Z + K(X,Z)FY - K(X,Y)FZ] , \end{aligned} \tag{2}$$

where $K(X,Y,Z)$, $K(X,Y)$ and K are the curvature tensor, the Ricci tensor and scalar curvature of the Riemannian space, $K^* = K_b^a F_a^b$ and

$$a = \frac{n-4}{\phi^2 - (n-4)^2}, \qquad b = \frac{\phi}{\phi^2 - (n-4)^2} \tag{3}$$

(4) $$\alpha_1 = \frac{n-2}{(n-2)^2-\phi^2}, \qquad \beta_1 = -\frac{\phi}{(n-2)^2-\phi^2}.$$

In [3] the product concircular curvature tensor is obtained:

(5) $$\begin{aligned} S(X,Y,Z) &= K(X,Y,Z) \\ &- (\alpha K+\beta K^*)[g(X,Z)y - g(X,Y)Z + g(X,FZ) - g(X,FY)FZ] \\ &- (\beta K+\alpha K^*)[g(X,FZ)Y - g(X,FY)Z + g(X,Z)FY - g(X,Y)FZ], \end{aligned}$$

where

(6) $$\alpha = \frac{n(2-n)-\phi^2}{(n^2-\phi^2)[(2-n)^2-\phi^2]}, \qquad \beta = \frac{2\phi(1-n)}{(n^2-\phi^2)[(2-n)^2-\phi^2]}.$$

In [4] the product conharmonic curvature tensor is defined as follows:

(7) $$\begin{aligned} W(X,Y,Z) &= K(X,Y,Z) \\ &+ a[K(X,Y)Z - g(X,Z)K(Y) + K(X,FY)FZ - g(X,FZ)K(FY) \\ &- K(X,Z)Y + g(X,Y)K(Z) - K(X,FZ)FY + g(X,FY)K(FZ)] \\ &- b[K(X,FY)Z - g(X,FZ)K(Y) + K(X,Y)FZ - g(X,Z)K(FY) \\ &- K(X,FZ)Y + g(X,FY)K(Z) - K(X,Z)FY + g(X,Y)K(FZ)]. \end{aligned}$$

The purpose of the present note is to find some relations connecting the tensors $C(X,Y,Z)$, $P(X,Y,Z)$, $S(X,Y,Z)$ and $W(X,Y,Z)$.

2. First we note that the contraction of (2) with respect to Z gives the zero tensor.

We transvect (2) by $^{-1}g(X,Y)$, where ^{-1}g is the conjugate tensor of g, and denote the obtained tensor by $P(Z)$. Then we have

$$P(Z) = (1+2\alpha_1)K(Z) + 2\beta_1 K(FZ) - (\alpha_1 K+\beta_1 K^*)Z - (\alpha_1 K^*+\beta_1 K)FZ$$

and

(8) $$\begin{aligned} P(X,Z) = (1+2\alpha_1)K(X,Z) + 2\beta_1 K(X,FZ) - (\alpha_1 K + \\ + \beta_1 K^*)g(X,Z) - (\alpha_1 K^*+\beta_1 K)g(X,FZ) \end{aligned}$$

where

$$P(X,Z) = g(X,P(Z)).$$

Now, we calculate the tensor

$$P(X,Y,Z) + A[P(X,Z)Y - P(X,Y)Z + P(X,FZ)FY - P(X,FY)FZ] + B[P(X,FZ)Y - P(X,FY)Z + P(X,Z)FY - P(X,Y)FZ],$$

where A and B are some constants. Taking into account (2) and (8), we find that this tensor can be expressed in the form

$$\begin{aligned} &K(X,Y,Z) + \\ &+ [\alpha_1 + A(1+2\alpha_1) + 2B\beta_1][K(X,Z)Y-K(X,Y)Z+K(X,FZ)FY- \\ &- K(X,FY)FZ]+[\beta_1+2A\beta_1+B(1+2\alpha_1)][K(X,FZ)Y-K(X,FY)Z + \\ &+ K(X,Z)FY-K(X,Y)FZ] - [(A\alpha_1+B\beta_1)K+K(A\beta_1+B\alpha_1)K^*] \\ &\qquad \cdot [g(X,Z)Y-g(X,Y)Z+g(X,FZ)FY-g(X,FY)FZ] \\ &-[(A\beta_1+B\alpha_1)K+(A\alpha_1+B\beta_1)K^*][g(X,FZ)Y- \\ &- g(X,FY)Z+g(X,Z)FY-g(X,Y)FZ] . \end{aligned}$$

If we determine the numbers A and B such that

$$\alpha_1 + A(1+2\alpha_1) + 2B\beta_1 = 0,$$
$$\beta_1 + 2A\beta_1 + B(1+2\alpha_1) = 0,$$

we get

$$(9) \qquad A = \frac{2(\beta_1^2 - \alpha_1^2)-\alpha_1}{(1+2\alpha_1)^2-4\beta_1^2}, \qquad B = -\frac{\beta_1}{(1+2\alpha_1^2)^2-4\beta_1^2} .$$

By straight forward calculations we find

$$(1+2\alpha_1)^2 - 4\beta_1^2 = \frac{n^2-\phi^2}{(n-2)^2-\phi^2} .$$

Since $n \neq \phi$, $n^2-\phi^2 \neq 0$. Since $p > 2$ and $q > 2$, $(n-2)^2-\phi^2 \neq 0$.

Also, by direct calculation, we obtain

$$A\alpha_1 + B\beta_1 = \alpha , \qquad A\beta_1 + B\alpha_1 = \beta .$$

Therefore, we find

$$(10)\qquad \begin{aligned} &P(X,Y,Z) + A\big[P(X,Z)Y-P(X,Y)Z+P(X,FZ)FY-P(X,FY)FZ\big]\\ &+ B\big[P(X,FZ)Y-P(X,FY)Z+P(X,Z)FY-P(X,Y)FZ\big] = S(X,Y,Z)\end{aligned}$$

where the numbers A and B have the values (9).

3. We contract (5) with respect to Z and denote by S(X,Y) the obtained tensor. Then we have

$$\begin{aligned} S(X,Y) = K(X,Y)&+\{[\alpha(n-2)+\phi\beta]K+[\beta(n-2)+\phi\alpha]K^*\}g(X,Y)\\ &+\{[\phi\alpha+(n-2)\beta]K+[\phi\beta+\alpha(n-2)]K^*\}g(X,FY) .\end{aligned}$$

Taking into account (6), we have

$$\alpha(n+2)+\phi\beta = -\frac{n}{n^2-\phi^2}, \qquad \alpha\phi+\beta(n-2) = \frac{\phi}{n^2-\phi^2} .$$

Therefore

$$(11)\qquad \begin{aligned} S(X,Y) = K(X,Y) &+ \frac{1}{n^2-\phi^2}(-nK+\phi K^*)g(X,Y)+\\ &+ \frac{1}{n^2-\phi^2}(\phi K-nK^*)g(X,FY) .\end{aligned}$$

Taking into account (5) and (11), we get

$$\begin{aligned} S(X,Y,Z) &+ a\big[S(X,Y)Z-g(X,Z)S(Y)+S(X,FY)FZ-g(X,FZ)S(FY)\\ &- S(X,Z)Y+g(X,Y)S(Z)-S(X,FZ)FY+g(X,FY)S(FZ)\big]\\ &- b\big[S(X,FY)Z-g(X,FZ)S(Y)+S(X,Y)FZ-g(X,Z)S(FY)\\ &- S(X,FZ)Y+g(X,FY)S(Z)-S(X,Z)FY+g(X,Y)S(FZ)\big]\\ = K(X,Y,Z) &+ a\big[K(X,Y)Z-g(X,Z)K(Y)+K(X,FY)FZ-g(X,FZ)K(FY)\\ &- K(X,Z)Y+g(X,Y)K(Z)-K(X,FZ)FY+g(X,FY)K(FZ)\big]\\ &- b\big[K(X,FY)Z-g(X,FZ)K(Y)+K(X,Y)FZ-g(X,Z)K(FY)\\ &- K(X,FZ)Y+g(X,FY)K(Z)-K(X,Z)FY+g(X,Y)K(FZ)\big]\\ &+ \Big[\Big(-\alpha+\frac{2(an+\phi b)}{n^2-\phi^2}\Big)K - \Big(\beta+\frac{2(a\phi+bn)}{n^2-\phi^2}\Big)K^*\Big]\big[g(X,Z)Y-g(X,Y)Z +\\ &+ g(X,FZ)FY-g(X,FY)FZ\big] + \Big[-\Big(\beta+\frac{2(a\phi+bn)}{n^2-\phi^2}\Big)K +\\ &+ \Big(-\alpha+\frac{2(an+b\phi)}{n^2-\phi^2}\Big)K^*\Big]\big[g(X,FZ)Y-g(X,FY)Z+g(X,Z)FY-g(X,Y)FZ\big].\end{aligned}$$

But

$$-\alpha + \frac{2(an+\phi b)}{n^2-\phi^2} = a\alpha_1 - b\beta_1$$

and

$$-(\beta + \frac{2(a\phi + bn)}{n^2-\phi^2}) = a\beta_1 - b\alpha_1$$

because of (3) and (6). Therefore it follows that:

$$\begin{aligned} S(X,Y,Z) &+ a[S(X,Y)Z-g(X,Z)S(Y)+S(X,FY)FZ-g(X,FZ)S(FY) \\ &- S(X,Z)Y+g(X,Y)S(Z)-S(X,FZ)FY+g(X,FY)S(FZ)] \\ &- b[S(X,FY)Z-g(X,FZ)S(Y)+S(X,Y)FZ-g(X,Z)S(FY) \\ &- S(X,FZ)Y+g(X,FY)S(Z)-S(X,Z)FY+g(X,Y)S(FZ)] \\ &= C(X,Y,Z). \end{aligned} \tag{12}$$

4. Taking into account (5) and (11), we find:

$$\begin{aligned} & S(X,Y,Z)+\alpha_1[S(X,Z)Y-S(X,Y)Z+S(X,FZ)FY-S(X,FY)FZ] \\ & \quad +\beta_1[S(X,FZ)Y-S(X,FY)Z+S(X,Z)FY-S(X,Y)FZ] \\ = \ & K(X,Y,Z)+\alpha_1[K(X,Z)Y-K(X,Y)Z+K(X,FZ)FY-K(X,FY)FZ] \\ & \quad +\beta_1[K(X,FZ)Y-K(X,FY)Z+K(X,Z)FY-K(X,Y)FZ] \\ & \quad +[(-\alpha+\frac{\phi\beta_1-n\alpha_1}{n^2-\phi^2})K + (-\beta+\frac{\phi\alpha_1-n\beta_1}{n^2-\phi^2})K^*][g(X,Z)Y \\ & \quad -g(X,Y)Z+g(X,Z)FY-g(X,FY)FZ] + [(-\beta+\frac{\phi\alpha_1-n\beta_1}{n^2-\phi^2})K + \\ & \quad + (-\alpha+\frac{\phi\beta_1-n\alpha_1}{n^2-\phi^2})K^*][g(X,FZ)Y-g(X,FY)Z + \\ & \quad + g(X,Z)FY-g(X,Y)FZ] \ . \end{aligned}$$

But

$$\frac{\phi\beta_1-n\alpha_1}{n^2-\phi^2} = \alpha \quad \text{and} \quad \frac{\phi\alpha_1-n\beta_1}{n^2-\phi^2} = \beta$$

because of (6). Therefore it follows that:

$$\begin{aligned} S(X,Y,Z) &+ \alpha_1[S(X,Z)Y-S(X,Y)Z+S(X,FZ)FY-S(X,FY)FZ] \\ &+ \beta_1[S(X,FZ)Y-S(X,FY)Z+S(X,Z)FY-S(X,Y)FZ] \\ &= P(X,Y,Z) \ . \end{aligned} \tag{13}$$

5. Contracting (7) with respect to Z, we obtain

$$W(X,Y) = [1+a(n-4)-b\phi]K(X,Y)+[a\phi-b(n-4)]K(X,FY) + (aK-bK^*)g(X,Y)+(aK^*-bK)g(X,FY).$$

It is easy to see that

$$1+a(n-4)-b\phi = 0 \quad \text{and} \quad a\phi-b(n-4) = 0.$$

Therefore

$$W(X,Y) = (aK-bK^*)g(X,Y) + (aK^*-bK)g(X,FY).$$

Taking this into account, as well as (7), we have

$$W(X,Y,Z)+\alpha_1[W(X,Z)Y-W(X,Y)Z+W(X,FZ)FY-W(X,FY)FZ] +\beta_1[W(X,FZ)Y-W(X,FY)Z+W(X,Z)FY-W(X,Y)FZ]=W(X,Y,Z) +[(a\alpha_1-b\beta_1)K+(-b\alpha_1+a\beta_1)K^*][g(X,Z)Y-g(X,Y)Z+g(X,FZ)FY- - g(X,FY)FZ]+[(-b\alpha_1+a\beta_1)K+(a\alpha_1-b\beta_1)K^*][g(X,FZ)Y- - g(X,FY)Z+g(X,Z)FY-g(X,Y)FZ]$$

i.e.

$$\text{(14)} \quad W(X,Y,Z)+\alpha_1[W(X,Z)Y-W(X,Y)Z+W(X,FZ)FY-W(X,FY)FZ] +\beta_1[W(X,FZ)Y-W(X,FY)Z+W(X,Z)FY-W(X,Y)FZ] = C(X,Y,Z).$$

6. Taking into account (8) and (11), we can easily see that

$$\text{(15)} \qquad P(X,Y) = (1+2\alpha_1)S(X,Y) + 2\beta_1 S(X,FY)$$

and

$$\text{(16)} \qquad S(X,Y) = \frac{1+2\alpha_1}{(1+2\alpha_1)^2-4\beta_1^2}P(X,Y) - \frac{2\beta_1}{(1+2\alpha_1)^2-4\beta_1^2}P(X,FY).$$

7. Relations (10),(12),(13),(14),(15) and (16) are the required relations connecting the tensors C(X,Y,Z),P(X,Y,Z), S(X,Y,Z) and W(X,Y,Z).

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REZIME

JEDNA PRIMEDBA O PRODUKT-TENZORIMA KRIVINE

Dokazane su relacije (10),(12),(13),(14) i (15) koje povezuju sledeće tenzore: tenzor produkt-konforme krivine (1), tenzor produkt-projektivne krivine (2), tenzor produkt-koncirkularne krivine (5) i tenzor produkt-konharmonijske krivine (7). (7).