*Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu*
*knjiga 13 (1983)*
*Review of Research Faculty of Science-University of Novi Sad, Volume 13(1983)*

# AUTOPARALLEL CURVES OF RIEMANN-OTSUKI SPACES

*Djerdji F. Nadj*

*Prirodno-matematički fakultet. Institut za matematiku*
*21000 Novi Sad, ul. dr Ilije Djuričića br. 4, Jugoslavija*

**ABSTRACT**

In this paper we study a Riemann-Otsuki space $R\text{-}O_n$ and one of its m-dimensional $(m < n)$ subspaces, which is also a Riemann-Otsuki space, (see [1] and [2]). We denote that subspace by $R\text{-}O_m$. Our aim is to determine the conditions by which the autoparallel curves of $R\text{-}O_m$ are the autoparallel curves of $R\text{-}O_n$, too.

In [4] the author considers the autoparallel curves of Weyl-Otsuki spaces. Using the fact that the coefficients of the connection of the covariant and contravariant parts of Otsuki's spaces are different, he gives the autoparallel curves of the covariant and contravariant kind respectively. Following this way, we shall study autoparallel curves of the covariant kind in paragraph 1, and in paragraph 2 we shall consider autoparallel curves of the contravatiant kind. In paragraphs 3 and 4, we shall observe the above two kinds of autoparallel curves, especially if the subspace has an intrinsic or induced connection respectively.

---

*AMS Mathematics subject classification (1980): Primary 53B05; Secondary 53B15.*
*Key words and phrases: Riemann-Otsuki spaces, subspaces, autoparallel curves.*

PRELIMINARIES

The theory of Weyl-Otsuki spaces was laid down by A. Moór in [3]. We get $R\text{-}O_n$ spaces from $W\text{-}O_n$ spaces if we suppose that in the relation $\nabla_k g_{ij} = \gamma_k g_{ij}$ it holds that $\gamma_k = 0$. Namely, the $R\text{-}O_n$ space is an n-dimensional differentiable manifold with Riemannian metric tensor $g_{ij}$, $\det(g_{ij}) \neq 0$ and Otsuki's connection. The basic elements of the $R\text{-}O_n$ space are $g_{ij}$ and the tensor $P^i_j$, $\det(P^i_j) \neq 0$. As in [3] and [5] the invariant differential in the spaces of the Otsuki kind with the coordinates $x^i$ is defined by

$$(0.1) \qquad DT^i_j := P^i_a P^b_j \bar{D}T^a_b$$

where

$$(0.2) \qquad \bar{D}T^a_b := (\partial_k T^a_b + {}'\Gamma^a_{sk} T^s_b - {}''\Gamma^s_{bk} T^a_s)\, dx^k \; .$$

Tensor $P^i_j$ and the coefficients of connections ${}'\Gamma^i_{jk}$ and ${}''\Gamma^i_{jk}$ satisfy Otsuki's relation

$$(0.3) \qquad \partial_k P^i_j - {}'\Gamma^t_{jk} P^i_t + {}''\Gamma^i_{tk} P^t_j = 0 \; .$$

We suppose that the tensor $P^i_j$ has an inverse $Q^i_j$ and the relations

$$(0.4) \quad a) \quad P^i_j Q^j_s = \delta^i_s \; , \qquad b) \quad P^i_j g_{ia} = P^i_a g_{ij}$$

hold.

We define the subspace in $R\text{-}O_n$ by the relation

$$(0.5) \qquad x^i = x^i(u^1, \dots, u^m) \qquad (m < n) \; .$$

By our supposition $\operatorname{rank}(\partial x^i / \partial u^\alpha) = m$[1/], and we use the notation

$$(0.6) \qquad B^i_\alpha := \frac{\partial x^i}{\partial u^\alpha} \; .$$

The metric tensor of the subspace $R\text{-}O_m$ is defined as usually by

*1/. In this article Latin indices run from 1 to n and Greek indices $\alpha, \beta, \dots, \lambda$ run from 1 to m, but $\mu, \nu, \dots, \omega$ run from (m+1) to n.*

(0.7) $$G_{\alpha\beta} := g_{ij}B^i_\alpha B^j_\beta .$$

The basic tensor $P^\alpha_\beta$ of the subspace $R\text{-}O_m$ is defined by the projection of $P^i_j$ on the subspace and

(0.8) $$P^\alpha_\beta := P^i_j B^\alpha_i B^j_\beta$$

where

(0.9) $$B^\alpha_i := g_{ij}G^{\alpha\beta}B^j_\beta .$$

We define the inverse tensor of the tensor $P^\alpha_\beta$ by $\overset{*}{Q}{}^\alpha_\beta$, i.e.

(0.10) $$P^\alpha_\beta \overset{*}{Q}{}^\beta_\gamma = \delta^\alpha_\gamma .$$

As in the embedding space, in the subspace we define the invariant differential $\overset{*}{D}$ of the tensor $T^\alpha_\beta$ defined over the subspace by

(0.11) $$\overset{*}{D}T^\alpha_\beta := P^\alpha_\gamma P^\lambda_\beta \overset{*}{D}T^\gamma_\lambda$$

where

(0.12) $$\overset{*}{D}T^\gamma_\lambda := (\partial_\chi T^\gamma_\lambda + {}'\Gamma^\gamma_{\varepsilon\chi}T^\varepsilon_\lambda - {}''\overset{*}{\Gamma}{}^\varepsilon_{\lambda\chi}T^\gamma_\varepsilon)du^\chi .$$

We suppose that the tensor $G_{\alpha\beta}$ is a metric tensor of the Riemannian kind i.e. $\det(G_{\alpha\beta}) \neq 0$ and $\overset{*}{D}G_{\alpha\beta} = 0$. From this condition, using (0.12), we shall determine ${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ and by using Otsuki's relation analogous to (0.3) for $P^\alpha_\beta$, ${}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ and ${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ we get ${}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ (see [1]). We can determine the coefficients of the connections of the subspace in other ways, too. This will be seen in paragraph 4.

Using the tangent vectors $B^i_\alpha$ we can determine the vectors $N^\mu_i$ orthogonal to the subspace $R\text{-}O_m$ by the equations $B^i_\alpha N^\mu_i = 0$ and we get

(0.13) $$\delta^i_j = B^i_\alpha B^\alpha_j + N^i_\mu N^\mu_j .$$

It is known that if $m \neq n-1$ the vectors $N_i^{\mu}$ are not uniquely determined.

## 1. COVARIANT TYPE OF AUTOPARALLEL CURVES

We shall now consider the subspace $R\text{-}O_m$ defined by relation (0.5). The curve $C: u^{\alpha}(S)$ is an autoparallel curve of the subspace if the tangent vector $du^{\alpha}/ds$ [2] is a parallel displaced along C. Applying (0.11) and (0.12) in $\frac{\overset{*}{D}}{ds}(du^{\alpha}/ds) = 0$, contracting by $\overset{*}{Q}{}^{\alpha}_{\beta}$ and using (0.10) we get an equation of the autoparallel curve of a contravariant type in the form

$$(1.1) \qquad \frac{d^2u^{\alpha}}{ds^2} + \overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds} = 0 \;.$$

We ask under which conditions will the autoparallel curve of the observed type of the subspace be, at the same time, the autoparallel curve of this type in an embedding space, too.

Let

$$(1.2) \qquad C: x^i = x^i(u^{\alpha}(s))$$

be the autoparallel curve of the subspace $R\text{-}O_m$. Using the differential quotient of (1.2), applying (0.6) we get

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial u^{\alpha}}\frac{du^{\alpha}}{ds} = B^i_{\alpha}\frac{du^{\alpha}}{ds}$$

and

$$(1.3) \qquad \frac{d^2x^i}{ds^2} = \frac{\partial B^i_{\alpha}}{\partial u^{\beta}}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} + B^i_{\alpha}\frac{d^2u^{\alpha}}{ds^2} \;.$$

Since, according to our supposition, C is the autoparallel curve of the covariant type, eliminating $d^2u^{\alpha}/ds^2$ with (1.1) from (1.3) we get

---

2/ s always denotes the arc length as parameter.

$$\frac{d^2x^i}{ds^2} = \left( \frac{\partial B^i_\beta}{\partial u^\gamma} - B^i_\alpha \,{}'\!\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}(u(s))\right) \frac{du^\beta}{ds}\frac{du^\gamma}{ds} .$$

Hence if C is the autoparallel curve in the space $R\text{-}O_n$, it must be

$$(1.4) \qquad \left( \frac{\partial B^i_\beta}{\partial u^\gamma} + {}'\Gamma^i_{sk} B^s_\beta B^k_\gamma\right)\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = B^i_\alpha \,{}'\!\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} .$$

Now we can formulate.

THEOREM 1. *Relation (1.4) is a necessary and sufficient condition for curve C to be the autoparallel curve of the contravariant type on the subspace $R\text{-}O_m$ and in the embedding space $R\text{-}O_n$ too.*

P r o o f. It follows from the above condition, that the condition is sufficient, Now we shall prove that it is neces - sary, too. From (1.3) and the supposition that curve C is autoparallel in $R\text{-}O_n$, it follows that

$$-{}'\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = \frac{\partial B^i_\alpha}{\partial u^\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} + B^i_\alpha \frac{d^2u^\alpha}{ds^2}$$

or

$$B^i_\alpha \frac{d^2u^\alpha}{ds^2} = -\left( {}'\Gamma^i_{jk} B^j_\alpha B^k_\beta + \frac{\partial B^i_\alpha}{\partial u^\beta}\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds} .$$

Substituting (1.4) and contracting by $B^\delta_i$, we get (1.1) and curve C is autoparallel on $R\text{-}O_m$. It is obvious that (1.1),(1.3) and $d^2x^i/ds^2 + {}'\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$ do not hold at the same time if (1.4) does not hold.

After general theorem 1, we shall investigate same special cases.

THEOREM 2. *If (1.4) holds for curve C of the subspace $R\text{-}O_m$ and the vector $\xi^i = B^i_\alpha \xi^\alpha$ is a vector of the subspace defined along curve C in its direction, then along C it holds*

$$(1.5) \qquad \frac{\overset{*}{D}\xi^\alpha}{ds} = P^\alpha_\lambda B^\lambda_i Q^i_a \frac{D\xi^a}{ds} .$$

P r o o f. From (1.4) it follows that the considered curve is autoparallel in the subspace and in the embedding spase, too. Using definitions (0.2) and (0.6), the basic invariant differential quotient of $\xi^i$ in R-O$_n$ is

$$\frac{\bar{D}\xi^i}{ds} = \frac{d(B^i_\beta \xi^\beta)}{ds} + \Gamma^{\;\;i}_{sk} B^s_\alpha \xi^\alpha B^k_\beta \frac{du^\beta}{ds} \quad .$$

Multiplying by $g_{ij}B^j_\alpha$ we get

$$(1.6)\quad g_{ij}B^j_\alpha \frac{\bar{D}\xi^i}{ds} = g_{ij}B^j_\alpha \left[\frac{d\xi^\beta}{ds} B^i_\beta + \left(\frac{\partial B^i_\beta}{\partial u^\gamma} + \Gamma^i_{sk} B^s_\beta B^k_\gamma\right)\xi^\beta \frac{du^\gamma}{ds}\right] \quad .$$

According to the stipulation of the theorem, vector $\xi^\alpha$ satisfies

$$(1.7)\qquad \xi^\alpha = \xi \frac{du^\alpha}{ds}$$

and (1.4) holds. Substituting (1.7) and (1.4) we get

$$(1.8)\qquad g_{ij}B^j_\alpha \frac{\bar{D}\xi^i}{ds} = g_{ij}B^j_\alpha B^i_\beta \left[\frac{d\xi^\beta}{ds} + \overset{*}{\Gamma}{}^\beta_{\gamma\chi}\xi^\gamma \frac{du^\chi}{ds}\right] \quad .$$

Using definitions (0.7) and (0.12) we get

$$g_{ij}B^j_\alpha \frac{\bar{D}\xi^i}{ds} = G_{\alpha\beta}\frac{\overset{*}{\bar{D}}\xi^\beta}{ds} \quad .$$

Expressing the basic covariant differential quotient $\bar{D}/ds$ by the covariant differential quotient D/ds and contracting by $G^{\alpha\delta}$ we get

$$g_{ij}B^j_\alpha G^{\alpha\delta} Q^i_t \frac{D\xi^t}{ds} = \overset{*}{Q}{}^\beta_\gamma \frac{\overset{*}{D}\xi^\gamma}{ds} \quad .$$

Using definition (0.9) and contracting by $P^\alpha_\beta$ according to (0.10) we finally get (1.5).

Relation (1.5) means that the covariant differential of the contravariant vector in our subspace does not depend only on the projection of the covariant differential of the space R-O$_n$, but also on the tensor $P^\alpha_\beta$ of subspace R-O$_m$ and on tensor $Q^i_a$ which is the inverse of tensor $P^i_j$ of the R-O$_n$ space.

Instead condition (1.4) it is possible to take a stronger condition and formulate the following

THEOREM 3. *If in the subspace* $R\text{-}O_m$ *along* C *we suppose that condition*

$$B^i_\alpha \, {}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}(u) \frac{du^\gamma}{ds} = \Big[ \frac{\partial B^i_\beta}{\partial u^\gamma} + {}'\Gamma^i_{jk}(x) B^j_\beta B^k_\gamma \Big] \frac{du^\gamma}{ds} \tag{1.9}$$

*holds, then (1.5) holds for the optional vector* $\xi^\alpha$ *defined along* C.

P r o o f. Condition (1.9) is stronger than condition (1.4) and it follows from this that curve C is autoparallel in $R\text{-}O_m$ and $R\text{-}O_n$. A calculation analogous to the above gives (1.6). Using (1.9) we get (1.8). It is not difficult to see that this is identical to (1.5).

## 2. COVARIANT TYPE OF AUTOPARALLEL CURVES

Curves satisfying relation

$$\frac{D}{ds} \Big(g_{ij}(x) \frac{dx^j}{ds}\Big) = 0 \tag{2.1}$$

will be called autoparallel curves of a covariant type. In Riemannian spaces this equation is equivalent to relation $\frac{D}{ds}\Big( \frac{dx^j}{ds} \Big) = 0$ because $Dg_{ij} = 0$ and the Leibniz formula holds. Applying definition (0.1) and using the contraction by $Q^i_r$ according to (0.4), from (2.1) we get

$$\frac{dg_{rj}}{ds} \frac{dx^j}{ds} + g_{rj} \frac{d^2x^j}{ds} - {}''\Gamma_{rjk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

Multiplying by $g^{ir}$ and using the proposition that in $R\text{-}O_n$ spaces $\bar{D}g_{ij} = 0$, we get

$$\frac{d^2x^i}{ds^2} + {}''\Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \tag{2.2}$$

(see [4] (3.2) and (3.2a) with $\gamma_k = 0$). This is the equation of the autoparallel curve of the covariant type in R-$O_n$. The equation of the autoparallel curve of the covariant type in subspace R-$O_m$ is

$$(2.3)\qquad \frac{d^2u^\alpha}{ds} + {}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0,$$

because $\overset{*}{\bar{D}}G_{\alpha\beta} = 0$ and s is the arc lenght as parameter. Substituting $d^2u^\alpha/ds^2$ from(2.3) in (1.3) we get

$$\frac{d^2x^i}{ds^2} = \left(\frac{\partial B^i_\alpha}{\partial u^\beta} - B^i_\gamma {}''\overset{*}{\Gamma}{}^\gamma_{\alpha\beta}\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds}.$$

From this follows the condition of the covariant case, which is analogous to (1.4). This is

$$(2.4)\qquad \left(\frac{\partial B^i_\alpha}{\partial u^\beta} + {}''\Gamma^i_{jk}B^j_\alpha B^k_\beta\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds} = B^i_\gamma {}''\overset{*}{\Gamma}{}^\gamma_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}.$$

It is not difficult to see that the following holds.

THEOREM 4. *Condition (2.4) is sufficient and necessary so that the autoparallel curve of covariant type of the subspace should be the autoparallel curve of the embedding space, too.*

Now we shall study whether theorems analogous to theorems 2 and 3 of the first paragraph hold in this case, too. Applying definition (0.2) on vector $\xi_i$, which satisfies $\xi_i = B^\alpha_i\xi_\alpha$, and multiplying $\bar{D}\xi_i/ds$ by $g^{ij}B^\alpha_j$ we get

$$(2.5)\qquad g^{ij}B^\alpha_j\frac{\bar{D}\xi_i}{ds} = g^{ij}B^\alpha_j\left[\frac{d\xi_\beta}{ds}B^\beta_i + \left(\frac{\partial B^\beta_i}{\partial u^\chi} - {}''\Gamma^s_{ik}B^\beta_s B^k_\chi\right)\xi_\beta\frac{du^\chi}{ds}\right].$$

At first, using the definition of $B^\beta_i$ we galculate

$$g^{ij}B^\alpha_j\frac{\partial B^\beta_i}{\partial u^\chi} = g^{ij}B^\alpha_j\left[\frac{\partial g_{ir}}{\partial x^k}B^k_\chi B^r_\gamma G^{\gamma\beta} + \frac{\partial B^r_\gamma}{\partial u^\chi}g_{ir}G^{\gamma\beta} + g_{ir}B^r_\gamma\frac{\partial G^{\gamma\beta}}{\partial u^\chi}\right].$$

Using that $\bar{D}g_{ir} = 0$, the definition of $B^\beta_s$ and $B^\alpha_r B^r_\gamma = \delta^\alpha_\gamma$ we get

$$g^{ij}B^{\alpha}_{j}\left(\frac{\partial B^{\beta}_{i}}{\partial u^{\chi}} - {}''\Gamma^{s}_{ik}B^{k}_{\chi}B^{\beta}_{s}\right) = \left({}''\Gamma^{j}_{rk}B^{r}_{\gamma}B^{k} + \frac{\partial B^{j}_{\gamma}}{\partial u^{\chi}}\right) B^{\alpha}_{j}G^{\gamma\beta} + \frac{\partial G^{\alpha\beta}}{\partial u^{\chi}} \quad .$$

Substituting it in (2.5) using $G^{\alpha\beta}$, which we get from (0.7), we have

$$g^{ij}B^{\alpha}_{j}\frac{\bar{D}\xi_{i}}{ds} = G^{\alpha\beta}\frac{d\xi_{\beta}}{ds} + \left({}''\Gamma^{j}_{rk}B^{r}_{\gamma}B^{k}_{\chi} + \frac{\partial B^{j}_{\gamma}}{\partial u^{\chi}}\right) B^{\alpha}_{j}G^{\gamma\beta}\xi_{\beta}\frac{du^{\chi}}{ds} + \frac{\partial G^{\alpha\beta}}{\partial u^{\chi}}\xi_{\beta}\frac{du^{\chi}}{ds} \; .$$

Finally we add and subtract $G^{\alpha\beta}\,{}''\overset{*}{\Gamma}{}^{\lambda}_{\beta\chi}\xi_{\lambda}\frac{du^{\chi}}{ds}$ and we get

$$(2.6)\quad g^{ij}B^{\alpha}_{j}\frac{\bar{D}\xi_{i}}{ds} = G^{\alpha\beta}\frac{\bar{D}\xi_{\beta}}{ds} + \Big[\frac{\partial G^{\alpha\beta}}{\partial u^{\chi}} + {}''\overset{*}{\Gamma}{}^{\beta}_{\lambda\chi}G^{\alpha\lambda} + \Big({}''\Gamma^{i}_{jk}B^{j}_{\gamma}B^{k}_{\chi} + \frac{\partial B^{i}_{\gamma}}{\partial u^{\chi}}\Big)B^{\alpha}_{i}G^{\gamma\beta}\Big]\xi_{\beta}\frac{du^{\chi}}{ds} \quad .$$

Using the property that $\xi_{\beta}$ is a vector tangential to the observed curve C and $G^{\gamma\beta}\xi_{\beta} = \xi^{\gamma} = \xi\frac{du^{\gamma}}{ds}$ and using the proposition that condition (2.4) holds, from the above equation we get

$$(2.7)\quad g^{ij}B^{\alpha}_{j}\frac{\bar{D}\xi_{i}}{ds} = G^{\alpha\beta}\frac{\overset{*}{\bar{D}}\xi_{\beta}}{ds} + \left(\frac{\partial G^{\alpha\beta}}{\partial u^{\chi}} + {}''\overset{*}{\Gamma}{}^{\beta}_{\lambda\chi}G^{\alpha\lambda} + {}''\overset{*}{\Gamma}{}^{\alpha}_{\gamma\chi}G^{\gamma\beta}\right)\xi_{\beta}\frac{du^{\chi}}{ds} \quad .$$

It is known, that in Otsuki´s space it is possible to define the covariant and basic covariant differential with respect only to one of the coefficients of connections. We denote these differentials by $'D$ and $''D$ or $'\bar{D}$ and $''\bar{D}$ respectively. Hence we see that

$$\left(\frac{\partial G^{\alpha\beta}}{\partial u^{\chi}} + {}''\overset{*}{\Gamma}{}^{\beta}_{\lambda\chi}G^{\alpha\lambda} + {}''\overset{*}{\Gamma}{}^{\alpha}_{\gamma\chi}G^{\gamma\beta}\right)\frac{du^{\chi}}{ds} = \frac{{}''\overset{*}{\bar{D}}G^{\alpha\beta}}{ds} \quad .$$

Since we know that in the observed space $\overset{*}{\bar{D}}G_{\alpha\beta} = {}''\overset{*}{\bar{D}}G_{\alpha\beta} = 0$ and $G_{\alpha\beta}G^{\beta\gamma} = \delta^{\gamma}_{\alpha}$, one can see that ${}''\overset{*}{\bar{D}}G^{\alpha\beta}/ds = 0$. Now from (2.7) it follows

$$(2.8)\quad g^{ij}B^{\alpha}_{j}\frac{\bar{D}\xi_{i}}{ds} = G^{\alpha\beta}\frac{\overset{*}{\bar{D}}\xi_{\beta}}{ds} \; .$$

Substituting $B^{\alpha}_{j}$ from (0.9) and contracting by $G_{\alpha\gamma}$, we get

$B^i_\gamma \frac{\bar{D}\xi_i}{ds} = \frac{\overset{*}{\bar{D}}\xi_\gamma}{ds}$. Using (0.1) and (0.4) or (0.11) and (0.10) respectively, the basic invariant differentials $\bar{D}$, $\overset{*}{\bar{D}}$ can be expressed by the differentials $D$, $\overset{*}{D}$ respectively and so $\frac{\bar{D}\xi_i}{ds} = Q^r_i \frac{D\xi_r}{ds}$ and $\frac{\overset{*}{\bar{D}}\xi_\gamma}{ds} = \overset{*}{Q}{}^\alpha_\gamma \frac{\overset{*}{D}\xi_\alpha}{ds}$. Finally from (2.8) we get

$$(2.9) \qquad \frac{\overset{*}{D}\xi_\alpha}{ds} = P^\gamma_\alpha B^i_\gamma Q^r_i \frac{D\xi_r}{ds}$$

and it is possible to formulate

THEOREM 4. *If in subspace* $R\text{-}O_m$, *(2.4) holds, vector* $\xi_\alpha$ *is a vector defined along curve* C *in its direction and* $\xi_r = B^\alpha_r \xi_\alpha$, *then along* C *(2.9) holds.*

The above theorem can also possible be formulated along C for all vectors of subspace $R\text{-}O_m$, but with a condition stronger than (2.4). This condition is

$$(2.10) \qquad B^i_\alpha {}''\overset{*}{\Gamma}{}^\alpha_{\beta\chi} \frac{du^\chi}{ds} = \left[ \frac{\partial B^i_\beta}{\partial u^\chi} + {}''\Gamma^i_{jk} B^j_\beta B^k_\chi \right] \frac{du^\chi}{ds} \quad .$$

Substituting the right side of (2.10) in (2.6) and using the fact that in our space $\frac{{}''\bar{D}G^{\alpha\beta}}{ds} = 0$, we get (2.8). So the following holds.

THEOREM 5. *From (2.10) it follows that for vector* $\xi_\alpha$ *of the subspace, components of which in the embedding space* $R\text{-}O_n$ *are* $\xi_i = B^\alpha_i \xi_\alpha$, *along the autoparallel curves of the subspace, (2.9) holds.*

## 3. SPECIAL CASES WITH AN INTRINSIC CONNECTION OF THE SUBSPACE

In article [1] the author gives the formulae of the coefficients of connections ${}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ and ${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$. In these formulae the coefficients of connections ${}'\Gamma^i_{jk}$ and ${}'\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$ or ${}''\Gamma^i_{jk}$ and ${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma}$

respectively are connected in a special way. Indeed, the coefficients of connection ${}'\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ and ${}''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ in this case are the coefficients of intrinsic connection of subspace $R\text{-}O_m$. Substituting ${}'\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ and ${}''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ in conditions (1.4) and (2.4) respectively, we get conditions equivalent to them, which we denote by $(1\overset{*}{.}4)$ and $(2\overset{*}{.}4)$ respectively.

At first we observe autoparallel curves of the contravatiant type. From [1] (24) we have that

$$ {}'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma} = \overset{*}{Q}{}^{\beta}_{\alpha} B^{\alpha}_{i} \left[ B^{j}_{\delta} B^{k}_{\gamma} P^{i}_{a}\, {}'\Gamma^{a}_{jk} - P^{a}_{j} B^{j}_{\delta} N^{\mu}_{a} N^{b}_{\mu}\, {}''\Gamma^{i}_{bc} B^{c}_{\gamma} - \right. $$
$$ \left. - P^{a}_{j} B^{j}_{\delta} N^{\mu}_{a} (\partial_{\gamma} N^{i}_{\mu}) + P^{i}_{j} B^{j}_{\delta\gamma} \right] . $$

Using transformation $u^{\alpha} \to u^{\alpha'}$ of the coordinates we see that ${}'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma}$ given in the above form change themselves in the following way

$$ (3.1) \qquad {}'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma} = {}'\overset{*}{\Gamma}{}^{\beta'}_{\delta'\gamma'} \frac{\partial u^{\beta}}{\partial u^{\beta'}} \frac{\partial u^{\delta'}}{\partial u^{\delta}} \frac{\partial u^{\gamma'}}{\partial u^{\gamma}} + \frac{\partial u^{\beta}}{\partial u^{\beta'}} \frac{\partial^{2} u^{\delta'}}{\partial u^{\gamma} \partial u^{\delta}} P^{i}_{b} B^{\alpha'}_{i} \overset{*}{Q}{}^{\beta'}_{\alpha'} B^{b}_{\delta'} . $$

This is the transformation form of the coefficients of connection iff

$$ P^{i}_{b} B^{\alpha'}_{i} \overset{*}{Q}{}^{\beta'}_{\alpha'} B^{b}_{\delta'} = \delta^{\beta'}_{\delta'} . $$

A contraction by $P^{\lambda'}_{\beta'} B^{\delta'}_{a}$ gives

$$ P^{i}_{b} B^{\lambda'}_{i} (\delta^{b}_{a} - N_{a} N^{b}_{\mu}) = P^{\lambda'}_{\delta'} B^{\delta'}_{a} . $$

Now it is possible formulate

THEOREM 6. *Condition*

$$ (3.2) \qquad P^{i}_{b} B^{\delta'}_{i} N^{b}_{\mu} = 0 $$

*is necessary and sufficient for coefficients* ${}'\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$, *which are given in (3.1), to be the coefficients of connection. In this case the formula*

$$ (3.3) \qquad {}'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma} = {}'\Gamma^{a}_{bc} B^{\beta}_{a} B^{b}_{\delta} B^{c}_{\gamma} + B^{\beta}_{a} B^{a}_{\delta\gamma} , \quad B^{a}_{\delta\gamma} := \frac{\partial}{\partial u^{\gamma}} B^{a}_{\delta} $$ *holds.*

Substituting (3.3) in (1.4) we get

$$\left(\frac{\partial B^i_\beta}{\partial u^\gamma} + {'\Gamma}^i_{sk} B^s_\beta B^k_\gamma\right) \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = B^i_\alpha ({'\Gamma}^a_{bc} B^\alpha_a B^b_\beta B^c_\gamma + B^\alpha_a B^a_{\beta\gamma}) \frac{du^\beta}{ds}\frac{du^\gamma}{ds} .$$

Using (0.13) it follows that

$$(3.4) \qquad N^i_\mu N^\mu_a ({'\Gamma}^a_{bc} B^b_\beta B^c_\gamma + B^a_{\beta\gamma}) \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0 .$$

This relation is now stronger than condition (1.4),and it holds.

THEOREM 7. *Condition (3.4) is sufficient, but not necessary for the autoparallel curve of the contravariant type of subspace* R-$O_m$ *to be at the same time the autoparallel curve of the contravariant type of the embedding* R-$O_n$ *space.*

P r o o f. Let the curve $C: u^\alpha = u^\alpha(s)$ be the autoparallel curve of R-$O_m$. Then substituting $d^2u/ds^2$ from (1.1) in (1.3) and using (3.3) we get

$$\frac{d^2x^i}{ds^2} = \frac{\partial B^i_\alpha}{\partial u^\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} - B^i_\alpha B^\alpha_a ({'\Gamma}^a_{bc} B^b_\beta B^c_\gamma + B^a_{\beta\gamma}) \frac{du^\beta}{ds}\frac{du^\gamma}{ds} .$$

According to relation (0.13) and $\frac{dx^j}{ds} = B^j_\alpha \frac{du^\alpha}{ds}$ we get

$$\frac{d^2x^i}{ds^2} = -{'\Gamma}^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} + N^i_\mu N^\mu_a ({'\Gamma}^a_{bc} B^b_\beta B^c_\gamma + B^a_{\beta\gamma}) \frac{du^\beta}{ds}\frac{du^\gamma}{ds} .$$

Applying (3.4) we get that the observed autoparallel curve of the subspace is at the same time the autoparallel curve of the contravariant type in the embedding R-$O_n$ space.

Now we shall consider the inverse question. Let the given curve be an autoparallel curve of the contravariant type in R-$O_n$, i.e.

$$(3.5) \qquad \frac{d^2x^i}{ds^2} + {'\Gamma}^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

hols. Multiplying (3.3) by $\frac{du^\delta}{ds}\frac{du^\gamma}{ds}$ and using $\frac{dx^j}{ds} = B^j_\alpha \frac{du^\alpha}{ds}$ we get

$$'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma}\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds} = '\Gamma^{a}_{bc}B^{\beta}_{a}\frac{dx^{b}}{ds}\frac{dx^{c}}{ds} + B^{\beta}_{a}B^{a}_{\delta\gamma}\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds} \quad .$$

Since (3.5) holds we can eliminate $'\Gamma^{a}_{bc}\frac{dx^{b}}{ds}\frac{dx^{c}}{ds}$ and using (1.3) we finally get (1.1). So we can formulate

COROLLARY 1. *The observed autoparallel curve of the contravariant type of the embedding space* R-$O_n$ *without new conditions is an autoparallel curve of subspace* R-$O_m$, *if it belongs to this subspace.*

The stipulation of the above theorem follows directly from the relation (3.3), too. A contraction of (3.3) by $B^{k}_{\beta}\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds}$ according to (0.13) gives

$$B^{k}_{\beta}\,'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma}\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds} = \left[('\Gamma^{k}_{bc}B^{b}_{\delta}B^{c}_{\gamma} + B^{k}_{\delta\gamma}) - N^{k}_{\mu}N^{\mu}_{a}('\Gamma^{a}_{bc}B^{b}_{\delta}B^{c}_{\gamma}+B^{a}_{\delta\gamma})\right]\frac{du^{\delta}}{ds}\frac{du^{\gamma}}{ds} ,$$

i.e. if $'\overset{*}{\Gamma}{}^{\beta}_{\delta\gamma}$ has the form (3.3) it is not sufficient that (1.4) is satisfied, (3.4) must be satisfied too.

Further we shall consider autoparallel curves of the covariant type. For $''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ we use the formula

$$(3.6) \qquad ''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma} = ''\Gamma^{i}_{jk}B^{\alpha}_{i}B^{j}_{\beta}B^{k}_{\gamma} + B^{\alpha}_{i}B^{i}_{\beta\gamma}$$

(see [1] (16)). The coefficients $''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ consruced in this way satisfy the condition necessary and sufficient for the covariant differential of the metric tensor $G_{\alpha\beta}$ of the subspace to be zero. Substituting (3.6) in (2.4), using (0.13) we get

$$(3.7) \qquad N^{\mu}_{r}N^{i}_{\mu}(''\Gamma^{r}_{ak}B^{a}_{\alpha}B^{k}_{\beta} + B^{r}_{\alpha\beta})\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0 \ .$$

Now we can formulate

THEOREM 8. *Condition (3.7) is a sufficient, but not necessary condition to be the autoparallel curve of the covariant type of subspace* R-$O_m$ *at the same time the autoparallel curve of the covariant type of the embedding* R-$O_n$ *space.*

COROLLARY 2. *The observed autoparallel curve of embedding space* $R\text{-}O_n$ *without new conditions is an autoparallel curve of subspace* $R\text{-}O_m$, *if it belongs to this subspace.*

The proofs are analogous with the proofs given by the contravariant type.

Now we use the notation

$$"H^{\nu}_{\chi\alpha} := P^{\varepsilon}_{\alpha}P^{\nu}_{\sigma}("\Gamma^{s}_{jk}B^{j}_{\varepsilon}B^{k}_{\chi} + B^{s}_{\varepsilon\chi})N^{\delta}_{s}$$

given in [3] (3.5). Contraction by $Q^{\alpha}_{\lambda}Q^{\sigma}_{\nu}$ and substitution of the term which we got in (3.7) gives

$$N^{i}_{\mu}Q^{\varepsilon}_{\alpha}Q^{\mu}_{\nu}"H^{\nu}_{\beta\varepsilon}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0. \tag{3.8}$$

We suppose that in our subspace, (3.2) is satisfied and from this it follows that $N^{i}_{\mu}Q^{\mu}_{\nu} = N^{a}_{\nu}Q^{i}_{a}$ (see [2] (1.8)).

Substituting it in (3.8) we get $Q^{\varepsilon}_{\alpha}Q^{i}_{a}N^{a}_{\nu}"H^{\nu}_{\beta\varepsilon}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0$. As it was proved in [3] $N^{a}_{\nu}"H^{\nu}_{\beta\varepsilon} = "\overset{*}{\nabla}_{\beta}B^{a}_{\varepsilon}$. It is known that in Otsuki's spaces $Q^{\varepsilon}_{\alpha}Q^{i}_{a}"\overset{*}{\nabla}_{\beta}B^{a}_{\varepsilon} = B^{i}_{\alpha\|\beta}$ holds and finally (3.8) has the form

$$B^{i}_{\alpha\|\beta}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0 \quad \text{or} \quad \frac{\tilde{D}B^{i}_{\alpha}}{ds}\frac{du^{\alpha}}{ds} = 0 .$$

## 4. SPECIAL CASES OF A SUBSPACE WITH AN INDUCED CONNECTION

The induced connection of the subspace $R\text{-}O_m$ can be determined in various ways. For example

A/ If we suppose that for the covariant vectors $\xi_{\alpha}$ of the subspace $R\text{-}O_m$ satisfying $\xi_i = B^{\alpha}_{i}\xi_{\alpha}$ we can define the invariant differential by

$$\tilde{D}\xi_{\alpha} := B^{i}_{\alpha}D\xi_{i} \tag{4.1}$$

then the coefficients of connection $"\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ have a form like coefficients of connection $"\overset{*}{\Gamma}^{\alpha}_{\beta\gamma}$ of the intrinsic connection which are given in (3.6). Using Otsuki´s relation to determine the coefficients of connection ${}'\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and substituting ${}'\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and ${}''\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ in $(\tilde{1}.4)$ and $(\tilde{2}.4)$ we get results which are the same as in the former paragraph. With $(\tilde{1}.4),(\tilde{2}.4)$ and $(\tilde{\tilde{1}}.4),(\tilde{\tilde{2}}.4)$ we shall quote the equations we get from (1.4),(2.4) if in place of ${}'\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$, ${}''\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}$ we use ${}'\tilde{\Gamma}^{\alpha}_{\beta\gamma}$, ${}''\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$, ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ respectively .

B/ If we suppose that for the contravariant vectors of the subspace satisfying $\xi^{i} = B^{i}_{\alpha}\xi^{\alpha}$ we define the covariant differential by

$$(4.2) \qquad \tilde{\tilde{D}}\xi^{\alpha} := B^{\alpha}_{i}D\xi^{i}$$

then we get the coefficients of connection ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ in the form

$$(4.3) \qquad {}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma} = {}'\Gamma^{i}_{jk}B^{\alpha}_{i}B^{j}_{\beta}B^{k}_{\gamma} + B^{\alpha}_{i}B^{i}_{\beta\gamma}$$

( [1] (26) and [2] (1.1)). This is equivalent to (3.3), and the contravariant case coincides with that observed in the former paragraph.

The coefficients of connection ${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ in the form

$$(4.4) \qquad {}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma} = P^{r}_{j}B^{j}_{\lambda}\overset{*}{Q}{}^{\lambda}_{\beta}({}''\Gamma^{a}_{rk}B^{\alpha}_{a}B^{k}_{\gamma} - B^{\alpha}_{r\gamma}) + P^{a}_{b}B^{\alpha}_{a}N^{b}_{\mu}({}''\Gamma^{s}_{jk}B^{j}_{\lambda}B^{k}_{\gamma} - B^{s}_{\lambda\gamma})N^{\mu}_{s}\overset{*}{Q}{}^{\lambda}_{\beta}$$

we get from $P^{\alpha}_{\beta}$ and ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ using Otsuki´s relation. Now the tensor $P^{\alpha}_{\beta}$ and the coefficients of connections ${}'\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ and ${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ satisfy Otsuki´s relation. Since we study the Riemann-Otsuki subspaces it must be that $\tilde{\tilde{D}}G_{\alpha\beta} = 0$. In [1] it was proved that form (3.6) of the coefficients of connection ${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ is necessary and sufficient for the metric tensor of the subspace to be a covariant constant. This means that the coefficients of connection ${}''\tilde{\tilde{\Gamma}}{}^{\alpha}_{\beta\gamma}$ from (4.4) can be used only in the special case in which (4.4) reduces on (3.6). But in these cases, for the auroparallel curves of covariant type the same holds as in the former paragraph for the curves of that type.

C/ If we suppose that for the covariant and contravariant vectors of the subspace $R\text{-}O_m$ satisfying $\xi_i = B_i^\alpha \xi_\alpha$ and $\xi^i = B_\alpha^i \xi^\alpha$ respectively the invariant differential is defined by (4.1) and (4.2) respectively, we get the coefficients of connections defined by (3.6) and (4.3) (see [1] (26) and [2] (1.1)). In this case the coefficients of connection $'\tilde{\Gamma}_{\beta\gamma}^\alpha$ and $''\tilde{\tilde{\Gamma}}_{\beta\gamma}^\alpha$ and the tensor $P_\beta^\alpha$ must satisfy Otsuki's relation, or as was proved in [2] it must be

$$(4.5)\quad P_r^i B_i^\alpha N_\mu^t ({}'\Gamma_{sk}^a B_\beta^s B_\gamma^k + B_{\beta\gamma}^a) N_a^\mu - P_b^i B_\beta^b N_i^\mu ({}''\Gamma_{rk}^s B_s^\alpha B_\gamma^k + B_{r\gamma}^\alpha) N_\mu^r = 0.$$

Relation (4.5) is sufficient for the subspace of the $R\text{-}O_n$ space to be a Riemann-Otsuki space with the coefficients of connection $'\tilde{\Gamma}_{\beta\gamma}^\alpha$ and $''\tilde{\tilde{\Gamma}}_{\beta\gamma}^\alpha$ and the basic tensor $P_\beta^\alpha$. From the above observation it obviously follows that the autoparallel curves of the co- or contravariant type of subspace $R\text{-}O_m$ are at the same time the autoparallel curves of the embedding space, if (3.7) and (3.4) are satisfied. Inversely the autoparallel curve of the co- or contravariant type of the embedding space $R\text{-}O_n$ is at the same time the autoparallel curve of the observed type of subspace $R\text{-}O_m$ if it belongs to this subspace.

## REFERENCES

[1] Dj.F.Nadj, *On subspaces of Riemann-Otsuki space, Publ.de l'Inst.Math. Beograd NS 30(44)(1981), 53-58.*

[2] Dj.F.Nadj, *On orthogonal spaces of the subspaces of a Riemann-Otsuki space, Zbornik rad.Prir.Mat.Fak., Novi Sad, 11(1981),201-208.*

[3] Dj.F.Nadj, *The Gauss', Codazzi's and Kühnes equations of $R\text{-}O_n$ spaces, Acta Math.Hung., Budapest , 44(3-4)(1984), 92-99.*

[4] A.Moór, *Otsukische Übertragung mit rekurrentem Masstensor, Acta Sci. Math. Szeged, 40(1978), 129-142.*

[5] A.Moór, *Über verschiedene geodätische Abweichungen in Weyl-Otsukische Räumen, Publ.Math., Debrecen, 28 (1981), 247-258.*

[6] T.Otsuki, *On general connections I.Math.J.Okayama Univ. 9 (1959-60), 99-164.*

Received by the editors September 29, 1983.

REZIME

AUTOPARALELNE KRIVE RIEMANN-OTSUKIJEVIH PROSTORA

U radu su posmatrane autoparalelne krive potprostora Otsukijevog prostora. Odredjeni su uslovi pod kojima su te krive autoparalelne krive i u okolnom prostoru. Pošto su koeficijenti koneksije ko- i kontravarijantnog dela koneksije posmatranih prostora različiti posebno ispitujemo autoparalelne krive kovarijantnog tipa i autoparalele kontravarijantnog tipa.