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# ON THE SPECTRUM OF MENDELSOHN n-TUPLE SYSTEMS

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ABSTRACT

In [6] Mendelsohn n-tuple systems (MnSs), which represent a generalization of Mendelsohn triple systems, were introduced and the spectrum of such systems investigated. In this paper we obtain some new results on the spectrum of MnSs using the results from [7].

$1^{\circ}$ In [3] Mendelsohn introduced a generalization of Steiner triple systems which he called cyclic triple systems. Such systems are now called Mendelsohn triple systems (MTSs). A cyclic triple is a collection t of three ordered pairs, none of which has equal coordinates, such that an element occurs as a first coordinate of an ordered pair iff it occurs as a second coordinate of an ordered pair in t. A MTS is a pair (S,T) where S is a finite nonempty set and T is a collection of cyclic triples of elements of S, such that every ordered pair of distinct elements of S is contained in exactly one triple of T. The number |S| is called the order of (S,T). The spectrum of MTSs is the set of all integers

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$q > 1$ such that $q \not\equiv 2 \pmod 3$ except $q = 6$. A MTS is equivalent to a quasigroup satisfying the identities $(xy)x = y$ (semisymmetric) and $x^2 = x$ (idempotent).

In [6] a generalization of MTSs called Mendelsohn n-tuple systems was defined.

Let S be a finite nonempty set, $n \geq 3$. A cyclic n-tuple is the set

$$\{(x_1^{n-1}),(x_2^n),(x_3^n,x_1),\ldots,(x_n,x_1^{n-2})\}$$

of n distinct ordered $(n-1)$-tuples of elements of S, among which there is no $(n-1)$-tuple the coordinates of which are all equal. (By $x_m^n$ we denote the sequence $x_m,x_{m+1},\ldots,x_n$. If $m > n$, then $x_m^n$ will be considered empty. The sequence $x,x,\ldots$ $\ldots,x$ (n times) will be denoted by $\overset{n}{x}$ .)

A Mendelsohn n-tuple system (MnS), $n \geq 3$, is a pair (S,T), where S is a finite nonempty set and t is a collection of cyclic n-tuples of elements of S, such that every ordered $(n-1)$-tuple of elements of S, the coordinates of which are not all equal, belongs to exactly one cyclic n-tuple of T. The number $|S|$ is called the order of the MnS (S,T).

In [6] the spectrum of MnSs for different values of n was considered. It was proved that if n and q are even numbers, then there is no MnS of order q. The spectrum of M4S was determined to be the set of all odd integers greater than 1 and the spectrum of M5S was also investigated and some properties of MnS described. In this paper we obtain some new results on the spectrum of MnS using the results from [7] .

2° An n-ary groupoid (n-groupoid) (Q,A) is called an n-guasigroup iff the equation $A(a_1^{i-1},x,a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in N = \{1,\ldots,n\}$.

An n-quasigroup is called idempotent iff for every $x \in Q$ $A(\overset{n}{x}) = x$.

An n-quasigroup $(Q,A)$ is called cyclic iff it satisfies the identity

$$A(A(x_1^n),x_1^{n-1}) = x_n \ .$$

An n-quasigroup is cyclic iff for every $i \in N_n$ and all $x_1^{n+1} \in Q$

$$A(x_1^n) = x_{n+1} \Longleftrightarrow A(x_{i+1}^n,x_1^{i-1}) = x_i \ .$$

Cyclic n-quasigroups are a generalization of semisymmetric quasigroups.

A quasigroup $(Q,\cdot)$ is said to be self-orthogonal iff for every pair $(a,b) \in Q^2$ the system $xy = a$, $yx = b$ has a unique solution. An n-quasigroup $(Q,A)$ is called self-orthogonal iff for every $(a_1^n) \in Q^n$ there exists a unique $(b_1^n) \in Q^n$ such that $A_i(b_1^n) = a_i$, $i=1,\dots,n$, where $A_1 = A$ and $A_i$ are defined by $A_i(x_1^n) = A(x_i^n,x_1^{i-1})$, $i=2,\dots,n$. In [1], [2], [4], [5], the spectrum of self-orthogonal semisymmetric quasigroups (SOSQs) was investigated and in [7] the spectrum of self-orthogonal cyclic n-quasigroups (SOCnQs), which generalize the concept of SOSQs to higher dimensions, was considered.

$3^o$ In [6] the following relation between idempotent cyclic n-quasigroups and M(n+1)S was established:

Let n+1 be a prime, then there exists an idempotent cyclic n-quasigroup of order q iff there exists a M(n+1)S of order q.

Now we shall prove that every finite SOCnQ is necessarily idempotent.

THEOREM 1. *If* $(Q,A)$ *is a finite* SOCnQ, *then* $(Q,A)$ *is idempotent.*

P r o o f. Let $(Q,A)$ be a finite SOCnQ and $a \in Q$ an arbitrary element. Then $A(\overset{n}{a}) = b$ implies $A_i(\overset{n}{a}) = b$ for all $i=2,\dots,n$, where $A_i$ are defined by $A_i(x_1^n) = A(x_i^n,x_1^{i-1})$, $i=2,\dots$ $\dots,n$. Hence the ordered n-tuple $(\overset{n}{a})$ is a solution of the system

$$(1) \qquad A_i(x_1^n) = b, \quad i=1,\dots,n ,$$

where $A_1 = A$. Since $(Q,A)$ is self-orthogonal, the solution $(\overset{n}{a})$ of system (1) is unique.

This holds for all $a \in Q$, hence for every $a \in Q$ there is an element $b \in Q$ such that $(\overset{n}{a})$ is the unique solution of system (1). The mapping $f : a \mapsto b$ is obviously injective and since $Q$ is finite, it is a bijection.

Let $a \in Q$ be an arbitrary element and $A(\overset{n}{a}) = b$. If $f^{-1}(a) = c$, then $A(\overset{n}{c}) = a$. $A$ is cyclic which implies

$$A(b,\overset{n-1}{a}) = a, \quad A(a,b,\overset{n-2}{a}) = a,\dots,A(\overset{n-1}{a},b) = a .$$

Hence

$$A_i(\overset{n}{c}) = A_i(b,\overset{n-1}{a}), \quad i=1,\dots,n ,$$

and from the self-orthogonality of $A$ it follows that $a = b = c$. So, we have proved that for every $a \in Q$ $A(\overset{n}{a}) = a$.

From the preceding theorem and the quoted connection between idempotent cyclic n-quasigroups and M(n+1)Ss, we get the result that if n+1 is prime then every SOCnQ of order $q$ defines a M(n+1)S of the same order. Hence, using the results on the spectrum of SOCnQ obtained in [7], we get the following two theorems.

THEOREM 2. *Let* $n \geq 3$ *be prime,* $p_1,\dots,p_m$ *primes and* $k_1,\dots,k_m$ *positive integers such that* $p_i^{k_i} \equiv 1 \pmod n$, $i=1,\dots,m$. *Then for arbitrary non-negative integers* $\alpha_i$, $i=1,\dots,m$, *there exists a* MnS *of order*

$$q = p_1^{k_1\alpha_1} \cdots p_m^{k_m\alpha_m} .$$

THEOREM 3. *Let* $n \geq 3$ *be prime and* $p_1,\dots,p_m$ *primes such that* $p_i > n$, $i=1,\dots,m$. *Then there are positive integers*

$s_1,\dots,s_m$, $1 \le s_i \le n-1$, $i=1,\dots,m$, *such that for all positive integers* $\alpha_i$, $i=1,\dots,m$, *there is a* MnS *of order*

$$q = p_1^{s_1\alpha_1} \dots p_m^{s_m\alpha_m} .$$

REMARK. In some cases the values of $s_i$ in the preceding theorem can be determined as degrees of irreducible factors of the polynomials which are defined in [7].

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REZIME

O SPEKTRU MENDELSONOVIH SISTEMA n-TORKI

Mendelsonovi sistemi n-torki, koji predstavljaju generalizaciju Mendelsonovih sistema trojki, su definisani i odredjene su neke vrednosti iz njihovog spektra u [6]. U ovom radu dobijeni su novi rezultati o spektru Mendelsonovih sistema n-torki koristeći rezultate iz [7].