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PARASTROPHY INVARIANT n-QUASIGROUPS

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ABSTRACT

An n-quasigroup (Q,A) is called a G-n-quasigroup iff $A=A^{\sigma}$ for all $\sigma\in G$, where G is a subgroup of the symmetric group of degree n+1 and A^{σ} is defined by:

$$A^{\sigma}(x_{\sigma 1},\ldots,x_{\sigma n})=x_{\sigma(n+1)} \text{ iff } A(x_1,\ldots,x_n)=x_{n+1}.$$

In the paper G-n-quasigroups are considered, and some of their properties described.

1° First we shall give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_m, x_{m+1},\ldots,x_n$ will be denoted by $\{x_i\}_{i=m}^{n}$ or by x_m^n . If m>n, then x_m^n will be considered empty.

An n-ary groupoid (n-groupoid) (Q,A) is called an n-quasigroup iff the equation $A(a_1^{i-1},x,a_{i+1}^{n})=b$ has a unique solution x for every a_1^n, $b\in Q$, and every $i\in N_n=\{1,\ldots,n\}$.

An n-quasigroup (Q,A) is isotopic to an n-quasigroup (Q,B) iff there exists a sequence $T=(\alpha_1^{n+1})$ of permutations of Q such that the following identity

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$$B(x_1^n) = \alpha_{n+1}^{-1} A(\{\alpha_i x_i\}_{i=1}^n)$$

holds. T is called an isotopism, B is an isotope of A, and by $A^T = B$ we denote that A is isotopic to B by T. T^{-1} is defined by $T^{-1} = (\{\alpha_i^{-1}\}_{i=1}^{n+1})$.

If (Q,A) is an n-quasigroup and $\sigma \in S_{n+1}$, where S_{n+1} is the symmetric group of degree n+1, then the n-quasigroup A^σ defined by

$$A^\sigma(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)} \Longleftrightarrow A(x_1^n) = x_{n+1}$$

is called a σ-parastrophe (or simply parastrophe) of A. If $\sigma, \tau \in S_{n+1}$, then $(A^\sigma)^\tau = A^{\sigma\tau}$ and

$$A(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)} \Longleftrightarrow A^\tau(\{x_{\sigma\tau i}\}_{i=1}^n) = x_{\sigma\tau(n+1)}.$$

If $T = (\alpha_1^{n+1})$ is an isotopism of A, then $(A^T)^\sigma = (A^\sigma)^{T^\sigma}$, where $T^\sigma = (\{\alpha_{\sigma i}\}_{i=1}^{n+1})$.

If (Q,A) is an n-quasigroup and $\sigma \in S_{n+1}$ such that $A = A^\sigma$, then σ is called an autoparastrophism of A. The set of all autoparastrophisms of A is a subgroup of S_{n+1} which will be denoted by $\Pi(A)$.

An n-quasigroup (Q,A) is called cyclic [3] iff for every $i \in N_n$ and all $x_1^{n+1} \in Q$

$$A(x_1^n) = x_{n+1} \Longleftrightarrow A(x_{i+1}^{n+1}, x_1^{i-1}) = x_i.$$

2° DEFINITION 1. *If* (Q,A) *is an* n-*quasigroup and* G *is a subgroup of* S_{n+1} *such that* $A = A^\sigma$ *for every* $\sigma \in G$, *then* A *is called a* G-n-*quasigroup*.

It is obvious that an n-quasigroup (Q,A) is a G-n-quasigroup iff $A = A^\sigma$ for all $\sigma \in \Gamma$, where Γ is a set of generators of the group G.

Some examples of G-n-quasigroups are:

1. Totally symmetric n-quasigroups are G-n-quasigroups with $G = S_{n+1}$.

2. Cyclic n-quasigroups, investigated in [3], are G-n--quasigroups, where G is the cyclic group generated by the cycle (12...n+1).

3. In [2] D.G.Hoffman has given a construction of a G-n-quasigroup (Q,A) of order mp, for every m>n, p≥2, and every subgroup $G \subseteq S_{n+1}$, such that $\Pi(A) = G$.

4. Let (Q,+) be an Abelian group such that $x+x \neq 0$ for every $x \neq 0$. If a ternary operation A is defined by

$$A(x_1,x_2,x_3) = x_1 + x_2 - x_3,$$

then (Q,A) is a G-3-quasigroup, where G is Klein's four-group {(1),(12)(34),(13)(24),(14)(23)}. It is easy to see that A is neither totally symmetric nor cyclic, and that there exist such G-3-quasigroups of every order > 2.

From the definition of a parastrophe, we get the following proposition.

Let (Q,A) be an n-quasigroup and $\sigma \in S_{n+1}$, $\sigma i = n+1$. $A = A^\sigma$ iff for all $x_1^n \in Q$

$$A(x_{\sigma 1},\dots,x_{\sigma(i-1)},A(x_1^n),x_{\sigma(i+1)},\dots,x_{\sigma n}) = x_{\sigma(n+1)} .$$

Consequently, every G-n-quasigroup can be defined as an n-quasigroup satisfying a system of identities.

PROPOSITION 1. *Let* (Q,A) *be an* n-*quasigroup and* G *a subgroup of* S_{n+1} . *A is a* G-n-*quasigroup iff for all* $\sigma \in \Gamma$ *and all* $x_1^n \in Q$

$$A(x_{\sigma 1},\dots,x_{\sigma(i-1)},A(x_1^n),x_{\sigma(i+1)},\dots,x_{\sigma n}) = x_{\sigma(n+1)},$$

where Γ *is a set of generators of* G *and* $i=\sigma^{-1}(n+1)$.

In the preceding proposition, of course, Γ can be replaced by G.

From Proposition 1 it follows that the direct product of G-n-quasigroups is a G-n-quasigroup, which gives the possibility of constructing new G-n-quasigroups from the given ones.

PROPOSITION 2. *If* (Q,A) *is a* G-n-*quasigroup and* $\tau\in S_{n+1}$ *is such that the group* G *is invariant under the inner automorphism induced by* τ, *then* A^{τ} *is a* G-n-*quasigroup.*

P r o o f. Since G is invariant under the automorphism $\sigma \mapsto \tau\sigma\tau^{-1}$, it follows that for every $\sigma_i\in G$ there exists $\sigma_j\in G$ such that $\tau\sigma_i\tau^{-1}=\sigma_j$. Hence $A^{\tau\sigma_i\tau^{-1}}=A^{\sigma_j}=A$, and $(A^{\tau})^{\sigma_i}=A^{\tau}$ for all $\sigma_i\in G$, which means that A^{τ} is a G-n-quasigroup.

COROLLARY. *If* A *is a* G-n-*quasigroup and* G *a normal subgroup of a group* $G_1\subseteq S_{n+1}$, *then every parastrophe* A^{τ}, $\tau\in G_1$, *is also a* G-n-*quasigroup.*

PROPOSITION 3. *Let* (Q,A) *be a* G-n-*quasigroup, where* $G=\Pi(A)$. *A parastrophe* A^{τ} *is a* G-n-*quasigroup iff* G *is invariant under the inner automorphisam induced by* τ.

P r o o f. If $\tau G\tau^{-1}=G$, then from Proposition 2 it follows that A^{τ} is a G-n-quasigroup.

Conversely, let A^{τ} be a G-n-quasigroup. Then for all $\sigma_i\in G$ $(A^{\tau})^{\sigma_i}=A^{\tau}$, that is, $A^{\tau\sigma_i\tau^{-1}}=A$. Since $G=\Pi(A)$, the only parastrophes which are equal to A are parastrophes induced by permutations from G. Hence $\tau\sigma_i\tau^{-1}\in G$ for all $\sigma_i\in G$.

Now we shall consider isotopes of G-n-quasigroups.

THEOREM 1. *Let an* n-*quasigroup* A *be isotopic to a* G-n-*quasigroup* B. *Then* A *is isotopic to the parastrophe* A^{σ} *for every* $\sigma\in G$, *and every parastrophe* A^{τ}, *where* τ *is a permutation such that* G *is invariant under the inner automorphism induced by* τ, *is an isotope of a* G-n-*quasigroup.*

P r o o f. B is a G-n-quasigroup, hence $B=B^{\sigma}$ for all $\sigma\in G$. Since the corresponding parastrophes of isotopic n-quasigroups are isotopic, it follows that A^{σ} is isotopic to $B=B^{\sigma}$ for all $\sigma\in G$.

If τ is such that G is invariant under the inner automorphism induced by τ, then Proposition 2 implies that B^{τ} is a G-n-quasigroup. Hence the corresponding parastrophe A^{τ} of A is an isotope of the G-n-quasigroup B^{τ}.

THEOREM 2. *Let* (Q,A) *be an n-quasigroup isotopic to its parastrophe* A^{σ} *by an isotopism* $T=(\alpha_1^{n+1})$, $A^T=A^{\sigma}$, *where* $\sigma\in S_{n+1}$, *and let* $(i_1\dots i_r)(j_1\dots j_s)\dots(k_1\dots k_t)$ *be the decomposition of* σ *into v disjoint cycles (where the cycles of length 1 are included). Then there exist permutations* $\theta_1,\dots\theta_v$ *of the set Q and an n-quasigroup* (Q,B) *which is isotopic to A, such that B is isotopic to* B^{σ} *by the isotopism*

$$(1,\dots,1,\theta_1^{-1}\alpha_{i_1}\alpha_{\sigma(i_1)}\dots\alpha_{\sigma^{r-1}(i_1)}\theta_1,1,\dots$$
$$\dots,1,\theta_2^{-1}\alpha_{j_1}\alpha_{\sigma(j_1)}\dots\alpha_{\sigma^{s-1}(j_1)}\theta_2,1,\dots$$
$$\dots,1,\theta_v^{-1}\alpha_{k_1}\alpha_{\sigma(k_1)}\dots\alpha_{\sigma^{t-1}(k_1)}\theta_v,1,\dots,1),$$

where there are at least $n+1-v$ *identity components, and at most one nonidentity component for every cycle of* σ. *The nonidentity component which corresponds to the cycle* $(i_1,\dots,i_r)$ *can be at any of the places* $i_1,\dots,i_r$ *and analogously for other cycles. If* (i_1) *is a cycle of length 1, then the corresponding nonidentity component is* $\theta_1^{-1}\alpha_{i_1}\theta_1$.

P r o o f. As in Theorem 5 from [3], let B be an arbitrary isotope of A, $B=A^S$, $S=(\beta_1^{n+1})$, and since $A^T=A^{\sigma}$ we have

$$(((B^{S^{-1}})^T)^{\sigma^{-1}})^S=B$$

which implies $B^{S^{-1}TS^{\sigma}}=B^{\sigma}$, where $S^{-1}TS^{\sigma}=(\{\beta_i^{-1}\alpha_i\beta_{\sigma i}\}_{i=1}^{n})$.

If we put

(1_1) $\beta^{-1}_{i_2}\alpha_{i_2}\beta_{\sigma(i_2)}=1,\dots,\beta^{-1}_{i_r}\alpha_{i_r}\beta_{\sigma(i_r)}=1,$

(1_2) $\beta^{-1}_{j_2}\alpha_{j_2}\beta_{\sigma(j_2)}=1,\dots,\beta^{-1}_{j_s}\alpha_{j_s}\beta_{\sigma(j_s)}=1,$

....

(1_v) $\beta^{-1}_{k_2}\alpha_{k_2}\beta_{\sigma(k_2)}=1,\dots,\beta^{-1}_{k_t}\alpha_{k_t}\beta_{\sigma(k_t)}=1,$

and take $\beta_{i_1},\beta_{j_1},\dots,\beta_{k_1}$ to be arbitrary permutations of Q, then solving the systems (1_1), $(1_2),\dots,(1_v)$ as it is done in [3], Theorem 5, the theorem follows.

THEOREM 3. *If an* n-*quasigroup* (Q,A) *is isotopic to an* n-*quasigroup* B *which coincides with one of its parastpophes,* $B=B^\sigma$, *then* A *is isotopic to* A^σ *by an isotopism* $T=(\alpha_1^{n+1})$ *such that*

$$\alpha_{i_1}\alpha_{\sigma(i_1)}\dots\alpha_{\sigma^{r-1}(i_1)}=1$$

for every cycle $(i_1\dots i_r)$ *in the decomposition of* σ *into disjoint cycles including cycles of length 1 (where, if* (j) *is a cycle of length 1, then* $\alpha_j=1$).

P r o o f. Let A be isotopic to n-quasigroup B, such that $B=B^\sigma$, by an isotopism $S=(\beta_1^{n+1})$, $A^S=B$. Then $A^S=B=B^\sigma=(A^S)^\sigma=(A^\sigma)^{S^\sigma}$, so $A^{S(S^\sigma)^{-1}}=A^\sigma$. As in [3], Theorem 6, denote $T=S(S^\sigma)^{-1}=(\alpha_1^{n+1})$ which implies

$$S^{-1}TS^\sigma=I=(1,\dots,1),$$

that is

$$\beta_k^{-1}\alpha_k\beta_{\sigma k}=1,\quad k=1,\dots,n+1. \tag{2}$$

If we solve the subsystems of (2) which correspond to the disjoint cycle decomposition of σ separately, as it is done in the preceding theorem, we shall have for the cycle $(i_1\dots i_r)$

$$\beta^{-1}_{i_1}\alpha_{i_1}\alpha_{\sigma(i_1)}\dots\alpha_{\sigma^{r-1}(i_1)}\beta_{i_1}=1,$$

or $\alpha_{i_1}\alpha_{\sigma(i_1)}\dots\alpha_{\sigma^{r-1}(i_1)}=1$, which completes the proof.

REMARK. Theorems 2 and 3 generalize some results on cyclic n-quasigroups from [3].

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REZIME

PARASTROFNO INVARIJANTNE n-KVAZIGRUPE

n-kvazigrupa (Q,A) se naziva G-n-kvazigrupa ako i samo ako je $A = A^\sigma$ za svako $\sigma \in G$, gde je G podgrupa simetrične grupe stepena n+1, a A^σ je definisana sa:
$A^\sigma(x_{\sigma 1},\dots,x_{\sigma n}) = x_{\sigma(n+1)}$ ako i samo ako je $A(x_1,\dots,x_n) = x_{n+1}$.

U ovom radu razmatrane su G-n-kvazigrupe i odredjena neka njihova svojstva.