*Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu*
*knjiga 13 (1983)*
*Review of Research Faculty of Science-University of Novi Sad, Volume 13(1983)*

# ALTERNATING SYMMETRIC n-QUASIGROUPS

*Zoran Stojaković*
*Prirodno-matematički fakultet. Institut za matematiku*
*21000 Novi Sad, ul.dr Ilije Djuričića br.4, Jugoslavija*

ABSTRACT

Alternating symmetric (AS) n-quasigroups are defined and considered. An n-quasigroup $(Q,f)$ is caled an AS-n-quasigroup iff $f(x_1,\dots,x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma 1},\dots,x_{\sigma n}) = x_{\sigma(n+1)}$ for every even permutation $\sigma$ of the set $\{1,\dots,n+1\}$. AS-n-quasigroups represent a generalization of semisymmetric quasigroups. Several equivalent definitions of an AS-n-quasigroup are given and it is proved that every AS-n-quasigroup, $n > 3$, defines a family of totally symmetric (n-2)-quasigroups. Some properties of (i,j)-associative AS-n-quasigroups are determined and full characterization of AS-n-groups is given. Autotopisms and isotopism of AS-n-quasigroups are considered. Necessary and sufficient conditions for a principal isotope of an AS-n-quasigroup to be an AS-n-quasigroup are given.

## 1°

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

---

*AMS Mathematics subject classification (1980): Primary 20N15; Secondary 20N05*
*Key words and phrases: n-quasigroup, n-group, parastrophism, isotopism.*

The sequence $x_m, x_{m+1}, \ldots, x_n$ we shall denote by $\{x_i\}_{i=m}^n$ or by $x_m^n$. If $m > n$, then $x_m^n$ will be considered empty. The sequence $x, x, \ldots, x$ (n times) will be denoted by $\overset{n}{x}$. If $n \le 0$, then $\overset{n}{x}$ will be considered empty.

An n-ary groupoid (n-groupoid) $(Q,f)$ is called an n-quasigroup iff the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution $x$ for every $a_1^n, b \in Q$ and every $i \in N_n = \{1,\ldots,n\}$.

An n-quasigroup $(Q,f)$ is isotopic to an n-quasigroup $(Q,g)$ iff there exists a sequence $T = (\alpha_1^{n+1})$ of permutations of $Q$ such that the following identity

$$g(x_1^n) = \alpha_{n+1}^{-1} f(\{\alpha_i x_i\}_{i=1}^n)$$

holds. $T$ is called an isotopism, $g$ is an isotope of $f$, and by $f^T = g$ we denote that $f$ is isotopic to $g$ by $T$. If $\alpha_{n+1}$ is the identity mapping, then $g$ is said to be a principal isotope of $f$. $T^{-1}$ is defined by $T^{-1} = (\{\alpha_i^{-1}\}_{i=1}^{n+1})$. If $T$ is an isotopism of $(Q,f)$ to itself, that is, $f^T = f$, then $T$ is called an autotopism of $f$.

By $S_n$ we denote the symmetric group of degree $n$ and by $A_n$ its alternating subgroup.

If $(Q,f)$ is an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup $f^\sigma$ defined by

$$f^\sigma(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)} \iff f(x_1^n) = x_{n+1}$$

is called a $\sigma$-parastrophe (or simply parastrophe) of $f$. If $\sigma, \tau \in S_{n+1}$, then $(f^\sigma)^\tau = f^{\sigma\tau}$ and

$$f(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)} \iff f^\tau(\{x_{\sigma\tau i}\}_{i=1}^n) = x_{\sigma\tau(n+1)} \,.$$

If $T = (\alpha_1^{n+1})$ is an isotopism of $f$, then $(f^T)^\sigma = (f^\sigma)^{T^\sigma}$, where $T^\sigma = (\{\alpha_{\sigma i}\}_{i=1}^{n+1})$.

If $(Q,f)$ is an n-quasigroup and $\sigma \in S_{n+1}$ such that $f = f^\sigma$, then $\sigma$ is called an autoparastrophism of $f$. The set of all autoparastrophism of $f$ is a subgroup of $S_{n+1}$

which will be denoted by $\Pi(f)$.

An n-quasigroup $(Q,f)$ is called totally symmetric (TS) iff $f^{\sigma} = f$ for every $\sigma \in S_{n+1}$.

An n-quasigroup $(Q,f)$ is called $(i,j)$-associative iff the following identity holds

$$f(x_1^{i-1}, f(x_i^{i+n-1}), x_{i+n}^{2n-1}) = f(x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-1}).$$

An n-quasigroup which is $(i,j)$-associative for all $i,j \in N_n$ is called an n-group.

$2^{\circ}$

DEFINITION. *An n-quasigroup* $(Q,f)$ *is called alternating symmetric (AS) iff for every* $\sigma \in A_{n+1}$

$$f = f^{\sigma} .$$

It is obvious that an n-quasigroup $(Q,f)$ is an AS-n-quasigroup iff $f = f^{\sigma}$ for all $\sigma \in \Gamma$, where $\Gamma$ is a generating set of the group $A_{n+1}$.

From the definition it follows that every TS-n-quasigroup is also an AS-n-quasigroup. But there are AS-n-quasigroups which are not TS, which follows from [2] where D.G.Hoffman has proved that for every $m > n$, $p \geq 2$, and every subgroup $G \subseteq S_{n+1}$ there exists an n-quasigroup $(Q,f)$ of order $mp$ such that $\Pi(f) = G$.

When $n = 2$ from the definition it follows that a quasigroup (binary) $(Q,\cdot)$ is AS iff $(\cdot) = (\cdot)^{(123)} = (\cdot)^{(132)}$, i.e. iff $xy = z \Leftrightarrow yz = x \Leftrightarrow zx = y$. These equivalences imply that $(Q,\cdot)$ is an AS quasigroup iff the identities

$$(1) \qquad y(xy) = x , \qquad (xy)x = y$$

hold.

A quasigroup satisfying the identities (1) is called semisymmetric, so binary AS quasigroups are in fact semisymmetric quasigroups. In [5] so-called cyclic n-quasigroups where introduced and such n-quasigroups are another generalization of semisymmetric quasigroups (an n-quasigroup $(Q,f)$ is cyclic iff the identity $f(f(x_1^n),x_1^{n-1}) = x_n$ holds or equivalently iff $f = f^\sigma$, $\sigma = (1,2,\ldots,n+1)$).

$3^\circ$

AS-n-quasigroups can be described as n-quasigroups satisfying certain systems of identities. If $(Q,f)$ is an n-quasigroup and $\sigma \in S_{n+1}$, $\sigma k = n+1$, then $f^\sigma = f$ iff for all $x_1^n \in Q$

$$f(\{x_{\sigma i}\}_{i=1}^{k-1}, f(x_1^n), \{x_{\sigma i}\}_{i=k+1}^{n}) = x_{\sigma(n+1)} .$$

So we have the following theorem.

THEOREM 1. *An n-quasigroup* $(Q,f)$ *is an AS-n-quasigroup iff for every* $\sigma \in \Gamma$ *and all* $x_1^n \in Q$

$$f(\{x_{\sigma i}\}_{i=1}^{k-1}, f(x_1^n), \{x_{\sigma i}\}_{i=k+1}^{n}) = x_{\sigma(n+1)} ,$$

*where* $\Gamma$ *is a set of generators of* $A_{n+1}$ *and* $k = \sigma^{-1}(n+1)$.

From Theorem 1 we get that the direct product of AS-n-quasigroups is also an AS-n-quasigroup and a subquasigroup of an AS-n-quasigroup is an AS-n-quasigroup.

Now we shall give explicitly some of the systems of identities defining AS-n-quasigroups. A well known generating set of $A_n$ is $\Gamma = \{(123),(124),\ldots,(12n)\}$, and from Theorem 1 we get the following corollary.

COROLLARY 1. *An n-quasigroup* $(Q,f)$ *is AS iff at least one of the following (eqivalent) systems of identities holds*

$$(2)\qquad \begin{cases} f(x_2,x_i,x_3^{i-1},x_1,x_{i+1}^n) = f(x_1^n), & i = 3,\ldots,n, \\ f(x_2,f(x_1^n),x_3^n) = x_1 , & \end{cases}$$

$$\Gamma_1 = \{(123),(124),\ldots,(1,2,n+1)\},$$

$$(3)\quad \begin{cases} f(f(x_1^n),x_2^{n-1},x_1) = x_n, \\ f(x_1,f(x_1^n),x_3^{n-1},x_2) = x_n, \\ f(x_1^2,f(x_1^n),x_4^{n-1},x_3) = x_n, \\ \dots\dots\dots\dots\dots\dots\dots \\ f(x_1^{n-2},f(x_1^n),x_{n-1}) = x_n, \end{cases}$$

$$\Gamma_2 = \{(n+1,n,1),(n+1,n,2),\dots,(n+1,n,n-1)\},$$

$$(4)\quad \begin{cases} f(x_n,x_2^{n-1},f(x_1^n)) = x_1, \\ f(x_1,x_n,x_3^{n-1},f(x_1^n)) = x_2, \\ f(x_1^2,x_n,x_4^{n-1},f(x_1^n)) = x_3, \\ \dots\dots\dots\dots\dots\dots\dots \\ f(x_1^{n-2},x_n,f(x_1^n)) = x_{n-1}, \end{cases}$$

$$\Gamma_3 = \{(n,n+1,1),(n,n+1,2),\dots,(n,n+1,n-1)\}.$$

If $(Q,f)$ is an n-quasigroup, then (see [1]) a parastrofe $f^{\pi_i}$, where $i \in N_n$, is defined by

$$f^{\pi_i}(x_1^{i-1},x_{n+1},x_{i+1}^n) = x_i \iff f(x_1^n) = x_{n+1} .$$

The operation $f^{\pi_i}$ is called i-th inverse operation for $f$. An AS-n-quasigroup can be defined also using inverse operations for $f$.

THEOREM 2. *An n-quasigroup* $(Q,f)$ *is AS iff*

$$f^{\pi_i \pi_j} = f$$

*for every* $i,j \in N_n$.

The next theorem shows that for $n > 3$ every AS-n-quasigroup defines a family of TS-(n-2)-quasigroups.

THEOREM 3. *Let* $(Q,f)$ *be an AS-n-quasigroup,* $n > 3$, *and* $a,b \in Q$ *arbitrary elements,* $i,j \in N_{n+1}$, $i \neq j$. *The (n-2)-quasigroup* $(Q,f)$ *defined by*

$$g(x_1^{n-2}) = x_{n-1} \iff f(x_1^{i-1},a,x_i^{j-2},b,x_{j-1}^{n-2}) = x_{n-1}$$

*is a TS-(n-2)-quasigroup.*

P r o o f. Since the alternating group $A_{n+1}$ is (n-1)-fold transitive permutation group, it follows that for each two ordered (n-1)-tuples of elements from $N_{n+1}$ there exists a permutation from $A_{n+1}$ which maps one of these (n-1)-tuples onto another. This means that the equality

$$f(x_1^{i-1},a,x_i^{j-2},b,x_{j-1}^{n-2}) = x_{n-1} ,$$

which has n-1 variables, remains valid if all variables are arbitrarily permuted (where a, b remain at their places), i.e. $f(x_1^{i-1},a,x_i^{j-2},b,x_{j-1}^{n-2}) = x_{n-1} \iff f(y_1^{i-1},a,y_i^{j-2},b,y_{j-1}^{n-2}) = y_{n-1}$, where $(y_1^{n-1})$ is an arbitrary permutation of $(x_1^{n-1})$. Hence

$$g(x_1^{n-2}) = x_{n-1} \iff g(y_1^{n-2}) = y_{n-1},$$

which means that $g$ is TS.

$4^\circ$

Since every cycle of odd length is an even permutation, we have the following proposition.

PROPOSITION 1. *If* $n$ *is even, then every AS-n-quasigroup is a cyclic n-quasigroup.*

From the preceding proposition and the results obtained in [6] for (i,j)-associative cyclic n-quasigroups, we get the following two theorems.

THEOREM 4. *Let* $(Q,f)$ *be an (i,j)-associative AS-n-quasigroup,* $n$ *even. Then* $f$ *is (i+m,j+m)-associative for every integer* $m$ *(where* $(i+m,j+m)$ *is reduced modulo* $n$*).*

THEOREM 5. *Let* $(Q,f)$ *be an $(i,j)$-associative AS-n-quasigroup, $n$ even, where $j-i$ is relatively prime to $n$. Then $f$ is an n-group.*

THEOREM 6. *Let* $(Q,f)$ *be an $(i,j)$-associative AS-n-quasigroup, $n$ odd. Then $f$ is $(i+m,j+m)$-associative, where $m$ is an arbitrary integer such that $1 \le i+m \le n$, $1 \le j+m \le n$.*

P r o o f. Let $f$ be $(i,j)$-associative. Then for all $x_1^{2n-1} \in Q$

$$f(x_1^{i-1}, f(x_i^{i+n-1}), x_{i+n}^{2n-1}) = f(x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-1}).$$

Let $i,j < n$. $f$ is AS, hence $f = f^{\sigma}$, where $\sigma = (12\ldots n)$, that is

$$f(x_{2n-1}, x_1^{i-1}, f(x_i^{i+n-1}), x_{i+n}^{2n-2}) = f(x_{2n-1}, x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-2}),$$

i.e. $f$ is $(i+1,j+1)$-associative. If $1 < i,j$, by an analogous procedure, using $\sigma^{-1}$ instead of $\sigma$, we get that $f$ is $(i-1,j-1)$-associative.

Hence $f$ is $(i+m,j+m)$-associative, where $m$ is an arbitrary integer such that $1 \le i+m \le n$, $1 \le j+m \le n$.

THEOREM 7. *Let* $(Q,f)$ *be an n-group. Then* $(Q,f)$ *is AS iff there exists an Abelian group* $(Q,+)$ *such that* $x = -x$ *for all* $x \in Q$*, and*

$$f(x_1^n) = \sum_1^n x_i + c,$$

*where $c$ is a fixed element from $Q$.*

P r o o f. Let $(Q,f)$ be an AS-n-group. Then by Hosszù theorem there exist a group $(Q,\cdot)$, its automorphism $\theta$ and an element $c \in Q$ such that

$$f(x_1^n) = x_1 \theta x_2 \theta^2 x_3 \ldots \theta^{n-1} x_n c,$$

where $\theta c = c$ and for all $x \in Q$ $\theta^{n-1}x = cxc^{-1}$. $f$ is AS, hence $f = f^{\sigma}$, where $\sigma = (1,2,n+1)$, and the following identity is valid

$$f(x_2, f(x_1^n), x_3^n) = x_1,$$

that is

$$x_2\theta(x_1\theta x_2\theta^2 x_3 \ldots \theta^{n-1}x_n c)\theta^2 x_3 \ldots \theta^{n-1}x_n c = x_1. \tag{5}$$

If we put in the preceding equality $x_i = e$, $i = 1,\ldots,n$, where $e$ is the unit of $(Q,\cdot)$, we get that $c^2 = e$. Now putting in (5) $x_i = e$, $i = 2,\ldots,n$ it follows $\theta x_1 = x_1$, i.e. $\theta$ is the identity mapping of $Q$. If in (5) we put $x_i = e$, $i = 1,3,\ldots,n$, we obtain $\theta^2 x_2 = x_2^{-1}$ which means that for all $x \in Q$ $x = x^{-1}$. Hence $(Q,\cdot)$ is an Abelian group and

$$f(x_1^n) = x_1 x_2 \ldots x_n c.$$

The converse part of the theorem is obvious.

Since the group $(Q,\cdot)$ such that $x = x^{-1}$ for all $x \in Q$ is of order $2^t$, $t \in N$, and for every $t \in N$ there exists such group of order $2^t$, we have the following corollary.

COROLLARY 2. *There exists a nontrivial* finite AS-n-group* $(Q,f)$ *of order* $q$ *iff* $q = 2^t$, $t \in N$.

5°

PROPOSITION 2. *If* $T = (\alpha_1^{n+1})$ *is an autotopism of an AS-n-quasigroup* $(Q,f)$ *and* $\sigma \in A_{n+1}$, *then* $T^{\sigma} = (\{\alpha_{\sigma i}\}_{i=1}^{n+1})$ *is also an autotopism of* $f$.

P r o o f. Since $T$ is an autotopism we have $f^T = f$, and since $f$ is AS it follows $f^{\sigma} = f$. Hence $f = (f^T)^{\sigma} = (f^{\sigma})^{T^{\sigma}} = f^{T^{\sigma}}$, that is, $T^{\sigma}$ is an autotopism of $f$.

---

*) *An n-group* $(Q,f)$ *is called trivial iff* $|Q| = 1$.

PROPOSITION 3. *Let* $\alpha, \beta$ *be permutations of a set* $Q$, $n > 2$. $(\alpha, \beta, {}^{n-1}\varepsilon)$ *is an autotopism of an AS-n-quasigroup* $(Q,f)$ *iff* $\beta = \alpha^{-1}$ *and for some* $i,j \in N_n$, $i \neq j$, *the identity*

$$(6) \qquad f(x_1^{i-1}, \alpha x_i, x_{i+1}^n) = f(x_1^{j-1}, \alpha x_j, x_{j+1}^n)$$

*holds (by* $\varepsilon$ *we denote the identity mapping of* $Q$*).*

P r o o f. Let $(\alpha, \beta, {}^{n-1}\varepsilon)$ be an autotopism of $(Q,f)$. By Proposition 2 for every $i,j \in N_n$, $i \neq j$, $({}^{i-1}\varepsilon, \alpha, {}^{n-1}\varepsilon, \beta)$ and $({}^{j-1}\varepsilon, \alpha, {}^{n-j}\varepsilon, \beta)$ are autotopism of $f$. Consequently

$$\beta f(x_1^n) = f(x_1^{i-1}, \alpha x_i, x_{i+1}^n) = f(x_1^{j-1}, \alpha x_j, x_{j+1}^n).$$

Putting in the preceding identity $\alpha^{-1}x_i$ instead of $x_i$, we get

$$f(x_1^n) = f(x_1^{i-1}, \alpha^{-1}x_i, x_{i+1}^{j-1}, \alpha x_j, x_{j+1}^n),$$

i.e. $({}^{i-1}\varepsilon, \alpha^{-1}, {}^{j-i-1}\varepsilon, \alpha, {}^{n-j+1}\varepsilon)$ is an autotopism of $f$, which implies that $(\alpha, \alpha^{-1}, {}^{n-1}\varepsilon)$ is also an autotopism of $f$. Two autotopisms which differ in only one component must be equal, hence $\alpha = \alpha^{-1}$.

Now let the identity (6) holds for some $i,j \in N_n$, $i \neq j$. Putting in (6) $\alpha^{-1}x_i$ instead of $x_i$, we get similarly as in the preceding part of the proof, that $(\alpha, \alpha^{-1}, {}^{n-1}\varepsilon)$ is an autotopism of $f$.

REMARK. It is easy to see that if the identity (6) holds for some $i,j \in N_n$, $i \neq j$, then it holds for every such $i,j$.

Let $(Q,f)$ be an n-quasigroup, $i \in N_n$. A permutation $\alpha$ of $Q$ is said to be i-inverse regular for $f$ iff $({}^{i-1}\varepsilon, \alpha, {}^{n-1}\varepsilon, \alpha^{-i})$ is an autotopism of $f$. A permutation which is i-inverse regular for $f$ for every $i \in N_n$ is called inverse regular for $f$. The set of all inverse regular permutations for $f$ will be denoted by $V(f)$ (see [1]).

If $(Q,f)$ is an AS-n-quasigroup, $n > 2$, then it is easy to see that if $\alpha$ is for some $i \in N_n$ i-inverse regular for $f$, then $\alpha$ is inverse regular for $f$.

Now we have the following corollary from Proposition 3.

COROLLARY 3. *If* $(\alpha,\beta,{}^{n-1}_{\ \ \varepsilon})$ *is an autotopism of an AS-n-quasigroup* $(Q,f)$, $n > 2$, *then* $\alpha$ *and* $\beta$ *are inverse regular permutations for* $f$.

PROPOSITION 4. *Let* $T = (\alpha_1^{n+1})$ *be an autotopism of an AS-n-quasigroup* $(Q,f)$.

*If* $n \geq 2$, *then for any* $i,j,k \in N_{n+1}$, $i \neq j \neq k \neq i$, $(\alpha_i^{-1}\alpha_j,\alpha_j^{-1}\alpha_k,\alpha_k^{-1}\alpha_i,{}^{n-2}_{\ \ \varepsilon})$ *or* $(\alpha_j^{-1}\alpha_k,\alpha_i^{-1}\alpha_j,\alpha_k^{-1}\alpha_i,{}^{n-2}_{\ \ \varepsilon})$ *is an autotopism of* $f$.

*If* $n > 3$ *then for all* $i,j \in N_{n+1}$, $i \neq j$, $(\alpha_i^{-1}\alpha_j,\alpha_j^{-1}\alpha_i,{}^{n-1}_{\ \ \varepsilon})$ *is an autotopism of* $f$.

P r o o f. If $T = (\alpha_1^{n+1})$ is an autotopism of $f$, $n \geq 2$, then $T^{-1}$ and $T^{\phi}$, $\phi = (i,j,k)$, $i,j,k \in N_{n+1}$, $i \neq j \neq k \neq i$, are also autotopisms of $f$. Then

$$T^{-1}T^{\phi} = ({}^{i-1}_{\ \ \varepsilon},\alpha_i^{-1}\alpha_j,{}^{j-i-1}_{\ \ \ \ \varepsilon},\alpha_j^{-1}\alpha_k,{}^{k-j-1}_{\ \ \ \ \varepsilon},\alpha_k^{-1}\alpha_i,{}^{n-k+1}_{\ \ \ \ \varepsilon})$$

is also an autotopism of $f$.

If $\sigma = (3k)(2j)(1i)$ and $\tau = (12)(3k)(2j)(1i)$, then one of these two permutations is even. If $\sigma$ is even, then $(T^{-1}T^{\phi})^{\sigma}$ is an autotopism of $f$, and if $\tau$ is even then $(T^{-1}T^{\phi})^{\tau}$ is an autotopism of $f$, hence the first part of the proposition is proved.

Now let $n > 3$ and let $S = (\alpha_i^{-1}\alpha_j,\alpha_j^{-1}\alpha_k,\alpha_k^{-1}\alpha_i,{}^{n-2}_{\ \ \varepsilon})$ be an autotopism of $f$ (the proof is analogous if $(\alpha_j^{-1}\alpha_k,\alpha_i^{-1}\alpha_j,\alpha_k^{-1}\alpha_i,{}^{n-2}_{\ \ \varepsilon})$ is an autotopism). Applying to $S$ the first part of this proposition, taking the first and any two of the last $n-3$ components of $S$, we get that $(\alpha_i^{-1}\alpha_j,\alpha_j^{-1}\alpha_i,{}^{n-1}_{\ \ \varepsilon})$ is an autotopism of $f$.

PROPOSITION 5. *Let* $n > 3$. *If* $T = (\alpha_1^{n+1})$ *is an autotopism of an AS-n-quasigroup* $(Q,f)$, *then for every* $\sigma \in S_{n+1}$ $T^\sigma = (\{\alpha_{\sigma i}\}_{i=1}^{n+1})$ *is also an autotopism of* $f$.

P r o o f. By the preceding proposition, for every $i,j \in N_{n+1}$, $i \neq j$, $S = (\alpha_i^{-1}\alpha_j, \alpha_j^{-1}\alpha_i, \overset{n-1}{\varepsilon})$ is an autotopism of $f$. Proposition 2 implies that

$$S^{(2j)(1i)(34)} = S^{(2j)(1i)} = (\overset{i-1}{\varepsilon}, \alpha_i^{-1}\alpha_j, \overset{j-i-1}{\varepsilon}, \alpha_j^{-1}\alpha_i, \overset{n-j+1}{\varepsilon})$$

is an autotopism of $f$. Multiplying this autotopism by $T$ from the left, we get that

$$(\alpha_1^{i-1}, \alpha_j, \alpha_{i+1}^{j-1}, \alpha_i, \alpha_{j+1}^{n+1}) = T^{(ij)}$$

is an autotopism of $f$. We have obtained that for every transposition $(ij)$ $T^{(ij)}$ is an autotopism of $f$, the set of all transpositions generates $S_{n+1}$, hence $T^\sigma$ is an autotopism for every $\sigma \in S_{n+1}$.

For $n > 3$ some properties of AS-n-quasigroups are very close to the corresponding properties of TS-n-quasigroups. Now we shall give several propositions describing such properties, but we shall omit the proofs since they can be given on the basis of the proved theorems and propositions analogously to the corresponding proofs for TS-n-quasigroups (see [1],[4]).

PROPOSITION 6. *If* $A_i(f)$ *denotes the group of* $i$-*th components of all autotopisms of an AS-n-quasigroup* $(Q,f)$, *then* $A_i(f)$ *coincides with* $A_j(f)$ *for all* $i,j \in N_{n+1}$. *This group we shall denote by* $A_o(f)$.

PROPOSITION 7. *If* $n > 3$ *and* $T = (\alpha_1^{n+1})$ *is an autotopism of an AS-n-quasigroup* $(Q,f)$, *then* $\alpha_i^{-1}\alpha_j$ *is inverse regular for* $f$ *for every* $i,j \in N_{n+1}$.

PROPOSITION 8. *Every autotopism* $T = (\alpha_1^{n+1})$ *of an AS-n-quasigroup* $(Q,f)$, $n > 3$, *can be represented in the form*

$$T = \alpha(\lambda_1,\dots,\lambda_n,\varepsilon) ,$$

*where* $\lambda_1,\dots,\lambda_n \in V(f)$.

PROPOSITION 9. *If a component of an autotopism of an AS-n-quasigroup* $(Q,f)$ *is* $\varepsilon$, $n > 3$, *then all other componets of that autotopism are inverse regular permutations for* $f$.

PROPOSITION 10. *If* $(Q,f)$ *is an AS-n-quasigroup,* $n > 3$, *then the set* $V(f)$ *of all inverse regular permutations for* $f$ *is an Abelian group.*

PROPOSITION 11. *Let* $(Q,f)$ *be an AS-n-quasigroup,* $n > 3$. *If a component of an autotopism of* $f$ *is inverse regular for* $f$, *then all components of that autotopism are inverse regular for* $f$.

PROPOSITION 12. *Let* $(Q,f)$ *be an AS-n-quasigroup,* $n > 3$. *For every permutation* $\alpha \in A_o(f)$ *there exists exactly one permutation* $\phi \in V(f)$, *such that* $(\overset{n}{\alpha},\alpha\phi)$ *is an autotopism of* $f$.

PROPOSITION 13. *If* $T = (\alpha_1^{n+1})$ *is an autotopism of an AS-n-quasigroup* $(Q,f)$ *such that* $\alpha_1^{n+1} \in V(f)$, *then*

$$\alpha_1\alpha_2\dots\alpha_n = \varepsilon .$$

THEOREM 8. *A principal isotope* $(Q,g)$ *of an AS-n-quasigroup* $(Q,f)$, $n > 3$, $f^T = g$, $T = (\alpha_1^n,\varepsilon)$, *is an AS-n-quasigroup iff all components of* $T$ *are inverse regular for* $f$.

P r o o f. Let $(Q,g)$, $g = f^T$, be an AS-n-quasigroup. Then

$$g(x_2,g(x_1^n),x_3^n) = x_1, \tag{7}$$

which gives

$$f(\alpha_1x_2,\alpha_2f(\{\alpha_ix_i\}_{i=1}^n),\{\alpha_ix_i\}_{i=3}^n) = x_1 ,$$

and, since $f$ as also AS-n-quasigroup, it follows

$$f(x_1,\alpha_1 x_2,\{\alpha_i x_i\}_{i=1}^{n}) = \alpha_2 f(\{\alpha_i x_i\}_{i=1}^{n}) .$$

Hence

$$f(x_1^n) = \alpha_2 f(\alpha_1 x_1,\alpha_2\alpha_1^{-1}x_2,x_3^n) ,$$

i.e. $(\alpha_1,\alpha_2\alpha_1^{-1},{}^{n-2}\varepsilon,\alpha_2)$ is an autotopism of $f$.

By Proposition 9 $\alpha_1$ and $\alpha_2$ are inverse regular for $f$. Using instead of (7) other identities which are satisfied by every AS-n-quasigroup, we get by an analogous procedure that $\alpha_3,\dots,\alpha_n$ are also inverse regular for $f$.

Conversely, let now an n-quasigroup $(Q,f)$ be isotopic to an AS-n-quasigroup $(Q,f)$, $f^T = g$, $T = (\alpha_1^n,\varepsilon)$, where all components of $T$ are inverse regular permutations for $f$.

Then

$$\alpha_n g(g(x_1^n),x_2^{n-1},x_1) = \alpha_n f(\alpha_1 f(\{\alpha_i x_i\}_{i=1}^{n}),\{\alpha_i x_i\}_{i=2}^{n-1},\alpha_n x_1) =$$

$$= f(f(x_1,\{\alpha_i x_i\}_{i=2}^{n}),\{\alpha_i x_i\}_{i=2}^{n-1},x_1) = f(f(y_1^n),y_2^{n-1},y_1) = y_n = \alpha_n x_n,$$

where we have used that $\alpha_n$ and $\alpha_1$ are inverse regular for $f$, that $f$ is an AS-n-quasigroup and we have put $y_1 = x_1$, $y_i = \alpha_i x_i$, $i = 2,\dots,n$.

Hence we have obtained the identity

$$g(g(x_1^n),x_2^{n-1},x_1) = x_n .$$

One can prove analogously that $g$ satisfies all other identities from (3), so $(Q,g)$ is an AS-n-quasigroup.

REMARK. Since the second part of the proof of the preceding theorem is valid for every $n \geq 2$, it follows that for every such $n$ a principal isotope of an AS-n-quasigroup $(Q,f)$ is an AS-n-quasigroup if all components of the isotopy are inverse regular for $f$.

REFERENCES

[1] Белоусов В.Д., n-арные квазигруппы, "Штиинца", Кишинев, 1972.

[2] Hoffman D.G., *On the spectrum of n-quasigroups with given conjugate invariant subgroup, J.Comb. Theory, Ser. A, 35 (1983), 98-99.*

[3] Paunić Dj., Stojaković Z., *Parastrophy invariant n-quasigroups, Univ. u Novom Sadu. Zbornik radova Prir.-mat. fak., ser.mat., 13 (1983), 251-257.*

[4] Сандик М.Д., Соколов Е.И., Тотально-симметричные n-квазигруппы, Мат. исследования, Кишинев, том III, вып.2, 1968, 170-182.

[5] Stojaković Z., *Cyclic n-quasigroups, Univ. u Novom Sadu, Zbornik radova Prir.-mat. fak.,ser.mat., 12 (1982), 407-415.*

[6] Stojaković Z., Paunić Dj., *On a class of n-groups, Proc. of the II Symp. n-ary structures, Varna, 1983 (to appear).*

Received by the editors August 7, 1984.

REZIME

ALTERNATIVNO SIMETRIČNE n-KVAZIGRUPE

U radu su definisane i razmatrane alternativno simetrične (AS) n-kvazigrupe. n-kvazigrupa (Q,f) se naziva AS-n-kvazigrupa ako i samo ako za svaku parnu permutaciju $\sigma$ skupa $\{1,\ldots,n+1\}$ važi $f(x_1,\ldots,x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma 1},\ldots,x_{\sigma n}) = x_{\sigma(n+1)}$. AS-n-kvazigrupe predstavljaju jednu generalizaciju polusimetričnih kvazigrupa. Date su neke ekvivalentne definicije AS-n-kvazigrupa i dokazano da svaka AS-n-kvazigrupa, $n > 3$, definiše familiju totalno simetričnih (n-2)-kvazigrupa. Odredjene su neke osobine (i,j)-asocijativnih AS-n-kvazigrupa i data potpuna karakterizacija AS-n-grupa. Zatim su razmatrane autotopije i izotopije AS-n-kvazigrupa. Dati su potrebni i dovoljni uslovi da glavni izotop AS-n-kvazigrupa bude AS-n-kvazigrupa.