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ON (k,n,q) - NETS

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ABSTRACT

(k,n)-Nets, $n \in N\setminus\{1\}$, $k \in N\setminus\{1,\dots,n\}$, represent a generalization of k-nets, $k \in N\setminus\{1,2\}$; namely (k,2)-nets are k-nets [8-9]. Finite (k,n)-nets of order $q \in N\setminus\{1\}$ are also called (k,n,q)-nets [7]. In this article a connection between k,n and q is established.

(k,n)-nets, $n \in N\setminus\{1\}$, $k \in N\setminus\{1,\dots,n\}$, represent a generalization of k-nets, $k \in N\setminus\{1,2\}$; namely, (k,2)-nets are k-nets [8-9]. F.Radó considered (4,3)-nets in [1-2]. (n+1,n)-nets were considered by R.Bauer in [3], and (k,n)-nets by A.S.Bektenov in [4-6]. V.D.Belousov and A.S.Bektenov considered (k,n)-nets in [7]. (k,n)-nets are connected with multiquasigroups [10-11]. Finite (k,n)-nets of order $q \in N\setminus\{1\}$ are also called (k,n,q) - nets [7]. Some connections between k,n and q [7],[11] are known. In this paper a connection between k,n and q is established.

Let T be a nonempty set of elements called *points*. Let a nonempty set B have as its elements some subsets of the set T, called *blocks*. Finally, let the sets $L_1,\dots,L_k$, $k \in N\setminus\{1,\dots,n\}$,

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be equivalence classes classifying the set $\mathcal{B}$. Then we say that $(T, L_1, \ldots, L_k)$ is a (k,n)-net iff the following properties hold:

nR1. The intersection of any n blocks belonging to different classes $L_{i_1}, \ldots, L_{i_n}$; $i_1, \ldots, i_n \in \{1, \ldots, k\}$ has exactly one element (point); and

nR2. Any point from T belongs to exactly one block of each class L_i, $i \in \{1,2,\ldots,k\}$ [1].

EXAMPLE. By superimposing the pictures 1_1 and 1_2 we obtain the picture of a (4,3)-net of order 2. We have:

$T = \{A,B,C,D,A',B',C',D'\}$, $L_1 = \{ABCD$ [2] $A'B'C'D'\}$,

$L_2 = \{BB'CC', AA'DD'\}$, $L_3 = \{AA'BB', CC'DD'\}$, and

$L_4 = \{A'BC'D, AB'CD'\}$.

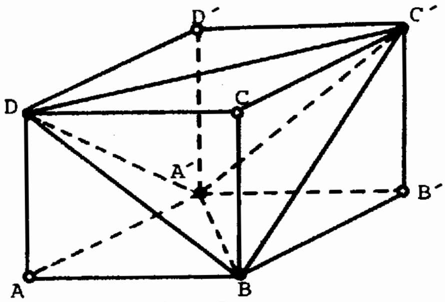

Fig. 1_1

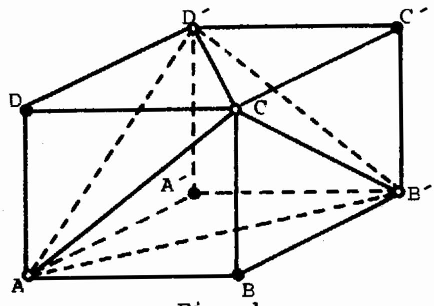

Fig. 1_2

Taking into account nR1, we can see that there does not exists a (k,3,2)-net for $k > 4$.

The following statement is known:

STATEMENT 1. All classes L_i, $i \in \{1,\ldots,k\}$, have the same number of blocks, and all blocks have the same number of points. Also, if $|L_i| = |Q|$ and $b_i \in L_i$, then $|b_i| = |Q^{n-1}|$.

The number of blocks in each class is called the order of (k,n)-net. A consequence of Statement 1 is:

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1) In the description of (k,n)-nets, the incidence relation is usually used.

2) We write ABCD instead of {A,B,C,D}.

STATEMENT 2. If $(T,L_1,\dots,L_k)$ is a (k,n)-net of order $q \in N\setminus\{1\}$, then the number of points in each block is q^{n-1}.

We are going to prove the following theorem.

THEOREM 3. *If* $(T,L_1,\dots,L_k)$ *is a* (k,n)- *net of order* $q \in N\setminus\{1\}$, *then*

(1) $$(k-1)\cdot\ldots\cdot(k-n+1) \leq (n-1)!q^{n-1}$$

REMARK. If n=2, (1) becomes $k-1 \leq q$, which is a known connection between k and q for k-nets of order q [8-9].

P r o o f. Let $b_i \in L_i$ and $T \notin b_i$ (Fig.2). There are exactly k blocks incident with T (nR2). Consider the blocks $b_1,\dots,b_{i-1},b_{i+1},\dots,b_k$ from the classes $L_1,\dots,L_{i-1},L_{i+1},\dots,L_k$ respectively. Each unordered (n-1)-tuple of blocks from the set $\{b_1,\dots,b_{i-1},b_{i+1},\dots,b_k\}$ has exactly one common point with the block $b_i \in L_i$ (nR1).

Let

$$B_\alpha = \{b_{\alpha_1},\dots,b_{\alpha_{n-1}}\}$$

and

$$B_\beta = \{b_{\beta_1},\dots,b_{\beta_{n-1}}\}$$

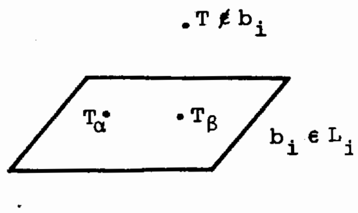

Fig. 2

be two such (n-1)-tuples; $|B_\alpha| = |B_\beta| = n-1$, and let

$$b_{\alpha_1} \cap \dots \cap b_{\alpha_{n-1}} \cap b_i = \{T_\alpha\} \in L_i \qquad \text{(Fig. 2) and}$$

$$b_{\beta_1} \cap \dots \cap b_{\beta_{n-1}} \cap b_i = \{T_\beta\} \in L_i \qquad \text{(Fig. 2)}$$

If $B_\alpha \neq B_\beta$, then $\max |B_\alpha \cup B_\beta| = 2n-2$ and $\min |B_\alpha \cup B_\beta| = n$. Then, in $B_\alpha \cup B_\beta$ there are at least n blocks; so, according to nR1, it follows that $T_\alpha \neq T_\beta$ provided that $B_\alpha \neq B_\beta$. Namely, if $T_\alpha = T_\beta$, then some n different blocks from $B_\alpha \cup B_\beta$ have two different common points - T and $T_\alpha = T_\beta$, but this is a contradiction with nR1.

So, the number of different common points of all the possible unordered $(n-1)$-tuples of blocks from the set $\{b_1,\ldots,b_{i-1},b_{i+1},\ldots,b_k\}$ with the block b_i equals the cardinality of the set

$$\{1,\ldots,k-1\}^{(n-1)}$$

i.e., this is the number $\binom{k-1}{n-1}$.

This number is not greater than the number of points in b_i, i.e., not greater than q^{n-1} (statement 2). So, it holds:

$$\binom{k-1}{n-1} \le q^{n-1} \quad ,$$

i.e., $(k-1)\cdot\ldots\cdot(k-n+1) \le (n-1)!\cdot q^{n-1}$.

REMARK. It can be found in [6] and [7] that

$$k \le (n-1)q + 1.$$

In [11] it is proved that

$$k \le n+q-1 \ .$$

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REZIME

O (k,n,q) - MREŽAMA

(k,n)- rešetke, $n \in N\backslash\{1\}$, $k \in N\backslash\{1,\dots,n\}$, predstavljaju jednu generalizaciju k-rešetaka, $k \in N\backslash\{1,2\}$; (k,2)-rešetke su, naime, k-rešetke [8-9]. Konačne (k,n)-rešetke reda $q \in N\backslash\{1\}$ zovemo i (k,n,q)-rešetke [7]. U ovom radu se utvrdjuje jedna veza izmedju k,n i q.