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SEMIGROUPS WHOSE PROPER IDEALS ARE ARCHIMEDEAN SEMIGROUPS

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ABSTRACT

In this paper semigroups whose proper (left) ideals are archimedean (left archimedean, t-archimedean, power joined) semigroups are considered.

In [10] T.E.Nordahl studied commutative Q-semigroups, i.e. commutative semigroups in which every proper ideal is power joined. C.S.H.Nagore, [8] extended Nordahl's results to quasi-commutative semigroups. A.Cherubini and A.Varisco in [6] considered Putcha's Q-semigroups. Weakly commutative semigroups in which every proper right ideal is power joined are studied by the author in [1]. B.Pondělíček, [12] considered uniform semigroups whose proper quasi-ideals are power joined. A characterization of Q-semigroups in the general case is given by A.Nagy, [9].

In the present paper we shall describe semigroups in which every proper two-sided ideal is an archimedean semigroup, (Theorem 1.) and in this way a generalization of the previous results is given. Theorem 1. is also a generalization of some results of [2,3,5]. Also, we shall decribe semigroups in which

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every proper left ideal is an archimedean (left archimedean, t-archimedean, power joined) semigroup, (Theorems 2,3,4,5.). At the end we describe semigroups in which every proper subsemigroup is power joined, (Theorem 6.).

Throughout this paper let N denote the set of all positive integers.

A semigroup S is archimedean if for any $a,b \in S$ there exists $n \in N$ for which $a^n \in SbS$, [11]. S is left archimedean if for every $a,b \in S$ there exists $n \in N$ such that $a^n \in Sb$, [13] (see also [14]). S is t-archimedean if for every $a,b \in S$ there exists $n \in N$ for which $a^n \in bS \cap Sb$, [13]. S is power joined if for every $a,b \in S$ there exist $m,n \in N$ such that $a^n = b^n$, [11]. S is special power joined if for every $a,b \in S$ there is an $n \in N$ such that $a^n = b^n$, [4].

Underfined notions and terminology are as in [7] and [11].

Let I(S)(L(S)) denote the union of all proper two-sided (left) ideals of a semigroup S.

THEOREM 1. *Every proper two-sided ideal of* S *is an archimedean subsemigroup of* S *if and only if* I(S) *is an archimedean subsemigroup of* S.

P r o o f. If all proper two-sided ideals of S are archimedean and $a,b \in I(S)$, then there is a proper two-sided ideal I of S with $a, aba \in I$ and there exists $n \in N$ such that

$$a^n \in IabaI \subseteq I(S)bI(S).$$

Thus I(S) is archimedean.

Conversely, let I(S) be archimedean and I be a proper two-sided ideal of S. Then for $a,b \in I$ there is an $n \in N$ such that $a^n = xby$ for some $x,y \in I(S)$. Hence $a^{n+2} = axbya$, where $ax, ya \in I$, and therefore I is archimedean.

LEMMA 1. *Let* L *be a (proper) left ideal of* S . *Then* L *is maximal if and only if*

(i) $S \setminus L = \{a\}, \quad a^2 \in L$

or

(ii) $S \setminus L \subseteq Sa$ *for every* $a \in S \setminus L$.

P r o o f. If L is a maximal left ideal of S, then we have the two cases: (i) There is an $a \in S \setminus L$ such that $Sa \subseteq L$. In this case $L \cup \{a\} = S$. Hence $S \setminus L = \{a\}$, $a^2 \in L$. (ii) For every $a \in S \setminus L$, $Sa \not\subseteq L$. Then $L \cup Sa = S$. Hence, $S \setminus L \subseteq Sa$ for every $a \in S \setminus L$.

The converse is obvious.

LEMMA 2. *Let* L(S) *be as in the case (ii) od Lemma 1. then*

$$S \setminus L(S) = \{x \in S : S = Sx\}$$

ia s subsemigroup of S.

P r o o f. For $a \in S \setminus L(S)$ we have that $S = L(S) \cup (S \setminus L(S)) = a \cup Sa$, so $L(S) \subseteq Sa$. From this and $S \setminus L(S) \subseteq Sa$ we have that $S = Sa$ for every $a \in S \setminus L(S)$. Conversely, let $S = Sa$ for every $a \in S \setminus L(S)$, then $S \setminus L(S) \subseteq Sa$, $a \in S \setminus L(S)$. Therefore $S \setminus L(S) = \{x \in S : S = Sx\}$ and it is clear that $S \setminus L(S)$ is a subsemigroup of S.

LEMMA 3. *Every left ideal of an archimedean (left archimedean,* t*-archimedean, power joined, special power joined) semigroup* S *is an archimedean (left archimedean,* t*-archimedean, power joined, special power joined) subsemigroup of* S.

P r o o f. Let L be an arbitrary left ideal of an archimedean semigroup S and $a, b \in L$. Then $a^n = xb^2y$ for some $x, y \in S$ and $n \in N$. It follows from this that $a^{n+1} = xbbya$ and $xb, ya \in L$.

THEOREM 2. *The following conditions are equivalent on a semigroup* S:

(1) *Every proper left ideal of* S *is archimedean;*

(2) L(S) *is archimedean;*

(3) S *satisfies one of the following conditions:*

(i) S *is archimedean;*

(ii) S *has a maximal left ideal* M *which is an archimedean semigroup and* $M \subseteq Ma$ *for any* $a \in S \setminus M$.

P r o o f. (1) => (2). If S is left simple, then S is archimedean. Assume that S is not left simple. If $a, b \in L(S)$, then there is a proper left ideal L of S such that $a, ba \in L$. Hence,

$$a^n \in LbaL \subseteq L(S)bL(S)$$

for some $n \in N$ and therefore L(S) is archimedean.

(2) => (3). If $L(S) \neq S$, then $M = L(S)$ is a maximal left ideal of S and by Lemma 1. we have that $S \setminus M = \{a\}$, $a^2 \in M$ or $S \setminus M \subseteq Sa$ for every $a \in S \setminus M$. If $S \setminus M = \{a\}$, $a^2 \in M$, then S is archimedean. If $S \setminus M \subseteq Sa$ for every $a \in S \setminus M$, then by Lemma 2. $T = S \setminus M$ is a subsemigroup of S. From $Sa = S$ $(a \in T)$ we have that $S = Ma \cup Ta \subseteq Ma \cup T \subseteq S$, i.e. $S = Ma \cup T$. Hence, $M \subseteq Ma$ for every $a \in S \setminus M$.

(3) => (1). If (i) holds, then by Lemma 3. every left ideal of S is archimedean. Let (ii) holds and let L be a proper left ideal of S. If $L \subseteq M$, then L is archimedean, (Lemma 3.). If $L \not\subseteq M$, then $L \cap (S \setminus M) \neq \emptyset$. For $a \in L \cap (S \setminus M)$ we have $M \subseteq Ma \subseteq L$, which is not possible.

THEOREM 3. *Every proper left ideal of a semigroup* S *is a left archimedean subsemigroup of* S *if and only if one the following conditions hold:*

1^o S *is left archimedean;*

2^o S *contains exactly two left ideals* L_1 *and* L_2 *which are left simple semigroup and* $S = L_1 \cup L_2$ *;*

3^o S *has a maximal left ideal* M *which is left archimedean and* $M \subseteq Ma$ *for every* $a \in S \setminus M$.

P r o o f. Let all proper left ideals of S are left archimedean. If $L(S) \neq S$, then $M = L(S)$ is a maximal left ideal of S which is left archimedean. By Lemma 1. we have $S \setminus M = \{a\}$, $a^2 \in M$ or $S \setminus M \subseteq Sa$ for every $a \in S \setminus M$. In the last case we have by Theorem 2. and Lemma 2. that $M \subseteq Ma$ for every $a \in S \setminus M$.

If $L(S) = S$ and for any two proper left ideals L_1, L_2 of S we have $L_1 \cap L_2 \neq \emptyset$, then S is left archimedean. Otherwise, there are left ideals L_1, L_2 of S with $L_1 \cap L_2 \neq \emptyset$. In this case $L_1 \cup L_2 = S$, since $L_1 \cup L_2$ is not left archimedean. Moreover, L_1 and L_2 are left simple semigroups and there exists no other proper left ideal L of S than L_1 and L_2. Consequently, if every proper left ideal of S is left archimedean, then we have one of the conditions 1°, 2° or 3°.

The converse follows immediately.

LEMMA 4.[1] *S is t-archimedean and left simple if and only if S is a group.*

THEOREM 4. *Let S not be left simple. Then every proper left ideal of S is t-archimedean if and only if one of the following conditions hold:*

1° *S is t-archimedean;*

2° *S contains exactly two left ideals G_1, G_2 which are groups and $S = G_1 \cup G_2$;*

3° *S has a maximal left ideal M which is a t-archimedean semigroup and $M \subseteq Ma$ for any $a \in S \setminus M$.*

P r o o f. Let every proper left ideal of S be t-archimedean. Then by Theorem 3. and Lemma 4. we have 2° or 3° or S is left archimedean. Assume that S is left archimedean. If $L(S) \neq S$, then $L(S)$ is a maximal left ideal of S and it is t-archimedean. By Lemmas 1. and 2. we have that $S \setminus L(S) = \{a\}$, $a^2 \in L(S)$ or $S \setminus L(S)$ is a subsemigroup of S. The last case is not possible and in the first case S is t-archimedean. If $L(S) = S$, then we can prove that S is of the type 1°.

The converse follows immediately.

The following theorem will be given without proof.

THEOREM 5. *Let S not be left simple. Then every proper left ideal of S is power joined if and only if one of the following conditions hold:*

1^o *S is power joined;*

2^o *S contains exactly two left ideals* G_1, G_2 *which are periodic groups and* $S = G_1 \cup G_2$ *;*

3^o *S has a maximal left ideal M which is power joined and* $M \subseteq Ma$ *for any* $a \in S \setminus M$.

THEOREM 6. *Every proper subsemigroup of S is power joined if and only* $|S| = 2$ *or S is power joined.*

P r o o f. Let S be not left simple. If any proper subsemigroup of S is power joined, then also any proper left ideal of S is power joined. Hence, by Theorem 5. we have one of the cases $1^o, 2^o$ or 3^o of this theorem. But, the cases 2^o and 3^o are possible only if $|S| = 2$. Indeed, let $S = G_1 \cup G_2$, where and G_2 are the disjoint left ideals of S which are periodic groups. If e and f are the units of G_1 and G_2, respectively, then it is clear that $S = \langle e,f \rangle$. Moreover, $ef \in G_2$ and $fe \in G_1$ and there exist $m,n \in N$ such that $f = (ef)^m$ and $e = (fe)^n$, so $ef = f$, $fe = e$. Therefore $S = \langle e,f \rangle = \{e,f\}$, i.e. $|S| = 2$. If we have 3^o, then $M = L(S)$, so $S \setminus M = \{a \in S : Sa = S\}$ is a subsemigroup of S (Lemma 2.). From $Sa = S$ $(a \in T = S \setminus M)$ we have $S = Ma \cup Ta$. If $Ma = M$, then we have $Ta = T$. Assume that T is not left simple. Then there is an element $a \in T$ with $Ta \nsubseteq T$. But, in this case $M \nsubseteq Ma$, hence, $Ma = S$. Let $a = xa$ for some $x \in M$. Then

$$(ax)^2 = a(xa)x = a^2x, \dots, (ax)^n = a^nx \in M \quad (n \in N)$$

and thus

$$\{ax, a^2x, \dots, a^nx, \dots\} \cup \{a, a^2, \dots, a^n, \dots\}$$

is a subsemigroup of S. Since this subsemigroup is not power joined it is equal to S and thus

$$M=\{ax,a^2x,\ldots\} \quad , \quad T=\{a,a^2,\ldots\} \quad .$$

Consequently, $x=a^kx$ for some $k\in N$. But then

$$a = xa = a^k(xa) = a^{k+1}$$

and T is a group. This is a contradiction. Therefore, T is a left simple semigroup and by Lemma 4. T is a subgroup of S. Let e be the unity of T. Then by 3^o we have $M\subseteq Me$ and thus for any $x\in M$ there is some $y\in M$ with $x=ye$. Hence, $xe=ye^2=$ $=ye=x$. For such an element $x\in M$ we have $(ex)^n=ex^n\in M$ $(n\in N)$ and $\{ex^2,ex^3,\ldots\}\cup\{e\}$ is a subsemigroup of S. This subsemigroup is not power joined, and thus it is equal to S. Consequently, $M=\{ex^2,ex^3,\ldots\}$, $T=\{e\}$. But in this case $ex=ex^k=(ex)^k$ for some $k>1$ and $M=\{ex,ex^2,\ldots\}$ is a group. For the unity $(ex)^{k-1}=ex^{k-1}$ of this group we have the subsemigroup $\{ex^{k-1},e\}$ of S which is not power joined. Hence, $S=\{ex^{k-1},e\}$, i.e. $|S|=2$.

Now, let S be left simple. Then we have two cases: (i) S is right simple. In this case S is a periodic group. (ii) If S is not right simple, then using the dual of Theorem 5. we have, as in the case that S is not left simple, that S is power joined or $|S|=2$.

The converse is obvious.

COROLLARY 1. [3] *Every proper subsemigroup of a semigroup S is special power joined if and only if* $|S|=2$ *or S is special power joined.*

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REZIME

PODGRUPE U KOJIMA SU SVI PRAVI IDEALI ARHIMEDOVSKE POLUGRUPE

U ovom radu razmatraju se polugrupe u kojima su svi pravi (levi) ideali arhimedovske (levo arhimedovske, t-arhimedovske, stepeno vezane) polugrupe.